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Generalized Order Statistics from a Novel Family of Distribution

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Abstract

In this article, we establish the moment's properties of a novel family of distribution are via generalized order statistics (GOS). We take into consideration the cases for order statistics and record values.

1 Introduction

In 1995, the probability distributions have been investigated by order statistics and record values separately. Kamps [1] proposed a model that unifies several models (order, sequential order, record, and Pfeifer's record) into one frame called GOS. The mechanism of this frame is arranged in ascending order of random variables.
Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed (IID) random variables (RV) based on distribution function (DF) $F(x)$ and probability density function (PDF) $f(x)$. Then the joint PDF of GOS is

$$f_{r,n,\tilde{m},k}(x_1, \cdots, x_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left[\bar{F}(x_n)\right]^{k-1} f(x_n) \left(\prod_{i=1}^{n-1} f(x_i) \left[\bar{F}(x_i)\right]^{m_i} \right). \tag{1.1}$$

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Moreover, we assume that $k \geq 1, n \geq 2$, and $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in R^{n-1}$ holds the relationship $\gamma_j = k + n - r + \sum_{j=r}^{n-1} m_j,\ 1 \leq r \leq n-1$ and $\bar{F}(x) = 1 - F(x)$. The marginal and joint density of GOS have been studied in two different contexts: $\gamma_i \neq \gamma_j$ and $\gamma_i = \gamma_j$. Throughout the paper, we consider $\gamma_i \neq \gamma_j$.
The PDF of r^{th} GOS is

$$f_{r,n,\tilde{m},k}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i - 1}. \tag{1.2}$$

The joint PDF of $(r-s)^{th}$ GOS is

$$f_{r,s,n,\tilde{m},k}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i} \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \tag{1.3}$$

where
$C_{r-1} = \prod_{i=1}^{r} \gamma_i$, $i = 1, 2, \cdots, n$, $a_i(r) = \prod_{j(\neq i)=1}^{r} \frac{1}{(\gamma_j - \gamma_i)}$, $\gamma_i \neq \gamma_j$, $1 \leq i \leq r \leq n$ and $a_i^{(r)}(s) = \prod_{j(\neq i)=r+1}^{s} \frac{1}{(\gamma_j - \gamma_i)}$, $\gamma_i \neq \gamma_j$, $r+1 \leq i \leq s \leq n$.
The recurrence relation based on GOS is a long-standing practice and an important issue in mathematical statistics just like others. In recent years, plenty of articles have dealt with the moments of generalized order statistics with several distributions. We refer the readers to [2, 3, 4, 5, 8, 9].

2 Kumaraswamy-Exponential Distribution

The Kumaraswamy-Exponential distribution (KED) is generated from the $T-X$ family proposed by [7]. Suppose $X_1, X_2, \cdots, X_n$ are the KED RV with three parameters α, β and γ. The CDF and PDF of KED are reported by

$$F(x) = 1 - \left[1 - (\alpha x)^{\beta}\right]^{\gamma},\ 0 < x < 1, \text{ and } \alpha, \beta, \gamma > 0 \tag{2.4}$$

$$f(x) = \alpha^{\beta} \beta \gamma x^{\beta - 1} \left[1 - (\alpha x)^{\beta}\right]^{\gamma - 1},\ x > 0, \tag{2.5}$$

respectively.
We note that the characterizing differential equation for KED from (2.5) and 2.4 is given by

$$\bar{F}(x) = \frac{1}{\alpha^{\beta} \beta \gamma} \left(x^{1-\beta} + \alpha^{\beta} x\right) f(x). \tag{2.6}$$

The KED has many applications in hydrology and industrial engineering and reduces to the Kumaraswamy distribution (KD) at $\alpha = 1$.
It has been aimed to establish the moment properties from the KED using GOS. Some recurrence relations for single and product moments are proved, which are not known in the existing literature.

3 Single Moments

The p^{th} moments of GOS for a random sample from $F(x)$ is

$$\mu^{p}_{r,n,\tilde{m},k} = C_{r-1} \int_{-\infty}^{\infty} x^p f(x) \sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i - 1} dx.$$

Theorem 3.1. *Let $X \sim KED(\alpha, \beta, \gamma)$ and $n \in N$, $k \geq 1$. Then the following relation holds for single moments:*

$$\mu^{p}_{r,n,\tilde{m},k} - \mu^{p}_{r-1,n,\tilde{m},k} = \frac{p}{\alpha^\beta \beta \gamma \gamma_r} \left[\mu^{p-\beta}_{r,n,\tilde{m},k} - \alpha^\beta \mu^{p}_{r,n,\tilde{m},k} \right]. \tag{3.7}$$

Proof: From [8], we have

$$\mu^{p}_{r,n,\tilde{m},k} - \mu^{p}_{r-1,n,\tilde{m},k} = pC_{r-2} \int_{0}^{\infty} x^{p-1} \sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i} dx. \tag{3.8}$$

Rewrite (3.8) as follows:

$$= \frac{p}{\alpha^\beta \beta \gamma} C_{r-2} \int_{0}^{1} x^{p-1} \sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_{i-1}} \left(x^{1-\beta} - \alpha^\beta x \right) f(x) dx. \tag{3.9}$$

Solving (3.9) confirms (3.7).

Corollary 3.2. *Replacing p by $-p$ in (3.7), we obtain the inverse moments of GOS for KED:*

$$\mu^{-p}_{r,n,\tilde{m},k} - \mu^{-p}_{r-1,n,\tilde{m},k} = \frac{-p}{\alpha^\beta \beta \gamma \gamma_r} \left[\mu^{-(p+\beta)}_{r,n,\tilde{m},k} - \alpha^\beta \mu^{-p}_{r,n,\tilde{m},k} \right].$$

Corollary 3.3. *At $\alpha = 1$, Theorem 3.1 reduces to single moments of KD :*

$$\mu^{p}_{r,n,\tilde{m},k} - \mu^{p}_{r-1,n,\tilde{m},k} = \frac{p}{\beta \gamma \gamma_r} \left[\mu^{p-\beta}_{r,n,\tilde{m},k} - \mu^{p}_{r,n,\tilde{m},k} \right].$$

as obtained in [9].

Remark 3.4. *(i) For order statistics ($m = 0$ and $k = 1$), the single moments from KED is* $E[X^p_{r:n}] - E[X^p_{r-1:n}] = \frac{p}{\alpha^\beta \beta\gamma\gamma_r} E[X^{p-\beta}_{r:n}] - \alpha^\beta E[X^p_{r:n}]$.

(ii) For the k^{th} record values ($k = -1$), the single moments from KED is

$$E(X^{(k)}_n)^p - E(X^{(k)}_{n-1})^p = \frac{p}{\alpha^\beta \beta\gamma k}\left[E(Y^{(k)}_n)^{p-\beta} - \alpha^\beta E(Y^{(k)}_n)^p\right].$$

4 Product Moments

The $(p, q)^{th}$ moments of GOS for a random sample from $F(x)$ is

$$\begin{aligned} \mu^{p,q}_{r,s:n,\tilde{m},k} &= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q C_{s-1}\left\{\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{\gamma_i}\right\} \\ &\quad \times \sum_{i=1}^{s} a_i(r)[\bar{F}(x_1)]^{\gamma_i}\frac{f(x_1)}{\bar{F}(x_1)}\frac{f(x_2)}{(\bar{F}(x_2)} dx_2 dx_1. \end{aligned}$$

Theorem 4.1. *As stated in Theorem 3.1, the following relation holds for product moment*

$$\mu^{p,q}_{r,s:n,\tilde{m},k} - \mu^{p,q}_{r,s-1:n,\tilde{m},k} = \frac{1}{\alpha^\beta \beta\gamma}\frac{q}{\gamma_s}\left[\mu^{p,q-\beta}_{r,s:n,\tilde{m},k} - \alpha^\beta \mu^{p,q}_{r,s:n,\tilde{m},k}\right]. \quad (4.10)$$

Proof: From [8], we have

$$\begin{aligned} \mu^{p,q}_{r,s:n,\tilde{m},k} - \mu^{p,q}_{r,s-1:n,\tilde{m},k} &= qC_{s-2}\int_0^1\int_{x_1}^1 x_1^p x_2^{q-1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{\gamma_i} \\ &\quad \times \sum_{i=1}^{s} a_i(r)[\bar{F}(x_1)]^{\gamma_i}\frac{f(x_1)}{\bar{F}(x_1)} dx_2 dx_1. \quad (4.11) \end{aligned}$$

$$\begin{aligned} &= qC_{s-2}\int_0^1\int_{x_1}^1 x_1^p x_2^{q-1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{\gamma_i} \\ &\quad \times \sum_{i=1}^{s} a_i(r)[\bar{F}(x_1)]^{\gamma_i}\frac{f(x_1)}{\bar{F}(x_1)}\frac{\bar{F}(x_2)}{\bar{F}(x_2)} dx_2 dx_1. \quad (4.12) \end{aligned}$$

Now, using (2.6) in (4.12), we obtain

$$\begin{aligned} \mu^{p,q}_{r:s,n,\tilde{m},k} - \mu^{p,q}_{r:s-1,n,\tilde{m},k} &= qC_{s-2}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)^{\gamma_i} \\ &\quad \times \sum_{i=1}^{s} a_i(r)[\bar{F}(x_1)]^{\gamma_i}\frac{f(x_1)}{\bar{F}(x_1)}\frac{f(x_2)}{\bar{F}(x_2)}\left\{x_2^{1-\beta} - \alpha^\beta x_2\right\} dx_2 dx_1, \quad (4.13) \end{aligned}$$

and so $\mu^{p,q}_{r,s:n,\tilde{m},k} = \mu^{p,q}_{r,s-1:n,\tilde{m},k} + \frac{1}{\alpha^{\beta}\beta\gamma}\frac{q}{\gamma_s}\left[\mu^{p,q-\beta}_{r,s:n,\tilde{m},k} - \alpha^{\beta}\mu^{p,q}_{r,s:n,\tilde{m},k}\right]$.

Remark 4.2. *(i) The relation for product moment based on order statistics for KED is* $\varphi^{p,q}_{r,s:n} - \varphi^{p,q}_{r,s-1:n} = \frac{1}{\alpha^{\beta}\beta\gamma}\frac{q}{(n-s+1)}\left[\varphi^{p,q-\beta}_{r,s:n} - \alpha^{\beta}\varphi^{p,q}_{r,s:n}\right]$.

(ii) The relation for product moment based on $K-th$ record values for KED is $\varphi^{p,q}_{k(r,s)} - \varphi^{p,q}_{k(r,s-1)} = \frac{1}{\alpha^{\beta}\beta\gamma}\frac{q}{k}\left[\varphi^{p,q-\beta}_{k(r,s)} - \alpha^{\beta}\varphi^{p,q}_{k(r,s)}\right]$.

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