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An inequality for Hermitian matrices

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Abstract

In this paper, we consider a certain inequality for positively definite Hermitian matrices. We prove this inequality using auxiliary inequalities such as Minkowski's inequality and the AM-GM inequality.

1 Introduction

In mathematics, inequalities have demonstrated their potency, notably in the realm of error analysis for filtering and estimation quandaries, adaptive stochastic control, and exploration of quantum mechanical Hamiltonians as evidenced by the work of Patel and Toda [1, 2, 3] as well as Lieb and Thirring [4]. In this paper, we establish novel and intriguing matrix and operator inequalities.

2 Main results

Theorem 2.1. *For a specific $n \times n$ Hermitian matrix A the following inequality holds:*

$$\det\left(\frac{1}{\lambda_{\max}}A + \lambda_{\max}A^{-1}\right) \geq \left(\frac{\mathrm{tr}A}{n\lambda_{\max}} + \frac{n\lambda_{\max}}{\mathrm{tr}A}\right)^n,$$

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where $\lambda_{\max}$ is the maximum eigenvalue of matrix A.

Proof.
Let's use the following lemma.
Lemma (Minkowski Inequality). *For any positively defined Hermitian matrices A and B the following inequality holds:*

$$(\det(A+B))^{\frac{1}{n}} \geq (\det A)^{\frac{1}{n}} + (\det B)^{\frac{1}{n}}.$$

Since the matrix A is Hermitian positive definite, it is obvious that A^{-1} is also Hermitian positive definite; i.e., $A \succ 0 \Rightarrow A^{-1} \succ 0$. Therefore, for matrices A and A^{-1}, the Minkowski inequality holds:

$$\left(\det\left(\frac{1}{\lambda_{\max}}A + \lambda_{\max}A^{-1}\right)\right)^{\frac{1}{n}} \geq \left(\det\left(\frac{1}{\lambda_{\max}}A\right)\right)^{\frac{1}{n}} + \left(\det\left(\lambda_{\max}A^{-1}\right)\right)^{\frac{1}{n}},$$

$$\det\left(\frac{1}{\lambda_{\max}}A + \lambda_{\max}A^{-1}\right) \geq \left\{\left(\det\left(\frac{1}{\lambda_{\max}}A\right)\right)^{\frac{1}{n}} + \left(\det\left(\lambda_{\max}A^{-1}\right)\right)^{\frac{1}{n}}\right\}^{n},$$

$$\det\left(\frac{1}{\lambda_{\max}}A + \lambda_{\max}A^{-1}\right) \geq \left\{\left(\frac{\det(A)}{\lambda_{\max}^{n}}\right)^{\frac{1}{n}} + \left(\lambda_{\max}^{n}\det\left(A^{-1}\right)\right)^{\frac{1}{n}}\right\}^{n},$$

$$\det\left(\frac{1}{\lambda_{\max}}A + \lambda_{\max}A^{-1}\right) \geq \left\{\frac{(\det(A))^{\frac{1}{n}}}{\lambda_{\max}} + \lambda_{\max}\left(\det\left(A^{-1}\right)\right)^{\frac{1}{n}}\right\}^{n}. \quad (2.1)$$

Given that the matrices A and A^{-1} are Hermitian, the decompositions $A = PDP^{-1}$ and $A^{-1} = PD^{-1}P^{-1}$ hold for them respectively. Thus inequality (2.1) can be expressed as:

$$\det\left(\frac{1}{\lambda_{\max}}D + \lambda_{\max}D^{-1}\right) \geq \left\{\frac{(\det(D))^{\frac{1}{n}}}{\lambda_{\max}} + \frac{\lambda_{\max}}{(\det(D))^{\frac{1}{n}}}\right\}^{n}$$

To see this, we will demonstrate that

$$\frac{(\det(D))^{\frac{1}{n}}}{\lambda_{\max}} + \frac{\lambda_{\max}}{(\det(D))^{\frac{1}{n}}} \geq \frac{\mathrm{tr}D}{n\lambda_{\max}} + \frac{n\lambda_{\max}}{\mathrm{tr}D}. \quad (2.2)$$

Obviously, from the AM-GM inequality, it follows that $\frac{\mathrm{tr}D}{n} \geq (\det(D))^{\frac{1}{n}}$. Inequality (2.2) can be rewritten as:

$$\left((\det(D))^{\frac{1}{n}} - \frac{\mathrm{tr}D}{n}\right)\left(\frac{\mathrm{tr}D}{n}(\det(D))^{-\frac{1}{n}} - \lambda_{\max}^{2}\right) \geq 0.$$

$\frac{trDn}{\geq}\left(\det\left(D\right)\right)^{\frac{1}{n}}$, $\left(\det\left(D\right)\right)^{\frac{1}{n}}-\frac{\mathrm{tr}D}{n}\leq 0$. Now, inequality (2.2) is equivalent to:

$$\frac{\mathrm{tr}D}{n}\left(\det\left(D\right)\right)^{\frac{1}{n}}-\lambda_{\max}^{2}\leq 0 \tag{2.3}$$

which is explicitly writted as:

$$\sqrt[n]{\lambda_1\cdot\lambda_2\cdot\lambda_3\cdot\ldots\cdot\lambda_n}\frac{\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n}{n}\leq\lambda_{\max}^{2},$$

$$\sqrt[n]{\frac{\lambda_1}{\lambda_{\max}}\cdot\frac{\lambda_2}{\lambda_{\max}}\cdot\frac{\lambda_3}{\lambda_{\max}}\cdot\ldots\cdot\frac{\lambda_n}{\lambda_{\max}}}\frac{\frac{\lambda_1}{\lambda_{\max}}+\frac{\lambda_2}{\lambda_{\max}}+\frac{\lambda_3}{\lambda_{\max}}+\ldots+\frac{\lambda_n}{\lambda_{\max}}}{n}\leq 1.$$

The last inequality is obviously true since $\frac{\lambda_k}{\lambda_{\max}}\leq 1\ \forall k\in\{1,\ldots,n\}$.

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