International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 1, (2025), 353–360
DOI: https://doi.org/10.69793/ijmcs/01.2025/edwin

Independence and equality of covariance matrices of two multivariate normal distributions

Johanna Orozco-Castañeda, Edwin Zarrazola, Daya K. Nagar

Instituto de Matemáticas
Universidad de Antioquia
Calle 67, No. 53–108
Medellín, Colombia

email: johanna.orozco@udea.edu.co, edwin.zarrazola@udea.edu.co,
dayaknagar@yahoo.com

(Received September 12, 2024, Accepted November 17, 2024,
Published November 21, 2024)

Abstract

The exact percentage points of the likelihood ratio statistic for testing the hypothesis that two p-variate normal distributions are independent and their covariance matrices are equal have been computed for $p = 2$, 3, 4 and 5.

1 Introduction

Suppose that the $2p \times 1$ random vector $\boldsymbol{X}$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ and that $\boldsymbol{X}$, $\boldsymbol{\mu}$ and Σ are partitioned as $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1' & \boldsymbol{X}_2' \end{pmatrix}'$, $\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1' & \boldsymbol{\mu}_2' \end{pmatrix}'$ and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, where $\boldsymbol{X}_i$ and $\boldsymbol{\mu}_i$ are $p \times 1$ and Σ_{ij} is $p \times p$, $i, j = 1, 2$. Consider testing the null hypothesis H that the subvectors $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent and covariance matrices of these sub-vectors are equal. That is, $H : \Sigma_{12} = 0, \Sigma_{11} = \Sigma_{22} = \Delta$ against the alternative H_a that H is not true. In H, the common covariance

Key words and phrases: Block sphericity, covariance matrix, distribution, test criterion, null moments, inverse Mellin transform, percentage points, residue theorem.
AMS (MOS) Subject Classifications: 62H10, 62H15.
ISSN 1814-0432, 2025, https://future-in-tech.net

matrix Δ is unspecified. While extending the circular symmetric model to the case where the symmetries are exhibited in blocks, Olkin [9] defined H and called it *block sphericity hypothesis*. Testing $\Sigma_{11} = \Sigma_{22} = \Delta$ under the assumption of independence of multivariate Gaussian distributions is commonly known as Bartlett's test and has been studied and applied in a variety of areas.

Let A be the sample sum of squares and product matrix formed from a sample of size $N = n+1$. Partition A as $A = (A_{ij})$, where A_{ij} is $p \times p$, $i, j = 1, 2$. The likelihood ratio statistic for testing H (Thomas [10], Cardeño and Nagar [3]) is given by

$$\Lambda = \frac{2^{pN} \det(A)^{N/2}}{\det\left(A_{11} + A_{22}\right)^{N}}.$$

The h^{th} null moment of $V = \Lambda^{1/N}$ is given as

$$E(V^h) = 2^{ph} \prod_{i=1}^{2p} \left\{ \frac{\Gamma[(h+n-i+1)/2]}{\Gamma[(n-i+1)/2]} \right\} \prod_{i=1}^{p} \left\{ \frac{\Gamma[n-(i-1)/2]}{\Gamma[h+n-(i-1)/2]} \right\}. \quad (1.1)$$

The exact non-null distribution of Λ under two specific alternatives has been derived by Gupta and Chao [5] . The asymptotic null distribution of $-2\ln\Lambda$ is chi-square with $p(3p+1)/2$ d.f. The null distribution of $\Lambda^{2/N}$, in series involving Bernoulli polynomials, is available in [4]. The exact null distribution of Λ is available in [3].

In this article, we compute the percentage points of the test statistic $V = \Lambda^{1/N}$ for testing H. Since the exact distribution is available in [3] and the technique is well known (Gupta, Nagar and Gómez [6], Nagar and Gupta [7], Nagar and Zarrazola [8], Zarrazola, Morán-Vásquez and Nagar [11]), we will outline main steps of the derivation and give the final result, deleting all the details of derivations. The exact percentage points of V for $p = 2(1)5$ are computed using the exact distribution given in this article.

2 The exact density of V

By using the duplication formula for gamma function the hth moment of V is rewritten as

$$E(V^h) = K(n,p) \prod_{i=1}^{p} \left\{ \frac{\Gamma(h+n-2p-1+2i)}{\Gamma[h+n-(i-1)/2]} \right\}, \quad (2.2)$$

where

$$K(n,p) = \prod_{i=1}^{p} \left\{ \frac{\Gamma[n-(i-1)/2]}{\Gamma(n-2p-1+2i)} \right\}. \tag{2.3}$$

Now, the density function of $V = \Lambda^{1/N}$, denoted by $f(v)$, is obtained by taking inverse Mellin transform of $E(V^h)$ as

$$f(v) = (2\pi\iota)^{-1} \int_L E(V^h) v^{-h-1}\, dh,\ 0 < v < 1, \tag{2.4}$$

where $\iota = \sqrt{-1}$ and L is a suitable contour. Using (2.2) in (2.4) and substituting $h + n - 2p = t$, one gets

$$f(v) = K(n,p) v^{n-2p-1} (2\pi\, \iota)^{-1} \int_{L_1} \left\{ \prod_{i=1}^{p} \frac{\Gamma(t+2i-1)}{\Gamma\left[t+2p-(i-1)/2\right]} \right\} v^{-t}\, dt, \tag{2.5}$$

where $0 < v < 1$, L_1 is the changed contour and $K(n,p)$ is defined in (2.3).

Next, we give explicit expressions for the density of V for particular values of p by evaluating the integral in (2.5) with the help of the residue theorem and simplifying resulting expressions by using properties of gamma, psi and zeta functions (Apostol [1], Askey and Roy [2]).

From (2.5), the density for $p = 2$ is obtained as

$$f(v) = K(n,2) v^{n-5} (2\pi\, \iota)^{-1} \int_{L_1} \frac{\Gamma(t+1)}{(t+3)\Gamma(t+7/2)} v^{-t}\, dt,\ 0 < v < 1.$$

The integrand has simple poles at $t = -r$, $r = 1, 2, 4, \ldots$, and a pole of order two at $t = -3$. Evaluating residues at these poles, simplifying resulting expressions and applying the residue theorem, the density $f(v)$ is obtained as

$$f(v) = K(n,2) v^{n-5} \left[-\frac{1}{\pi} \sum_{r=1(\neq 3)}^{\infty} \frac{\Gamma(r-5/2)}{(r-1)!\,(r-3)} v^r + \left\{ \frac{3}{2} - \ln\left(\frac{v}{4}\right) \right\} \frac{1}{2\sqrt{\pi}} v^3 \right], \tag{2.6}$$

where $0 < v < 1$. For $p = 3$, (2.5) simplifies to

$$f(v) = K(n,3) v^{n-7} (2\pi\iota)^{-1} \int_{L_1} \frac{\Gamma(t+1)}{(t+5)(t+4)(t+3)\Gamma(t+11/2)} v^{-t}\, dt,$$

where $0 < v < 1$. The integrand has simple poles at $t = -r$, $r = 1, 2, 6, 7, \ldots$, and poles of order two at $t = -r$, $r = 3, 4, 5$. Evaluating residues at these poles and using the residue theorem, the density in this case is obtained as

$$f(v) = \frac{K(n,3)}{\sqrt{\pi}} v^{n-7}\Bigg[\frac{2}{315}v - \frac{4}{45}v^2 - \frac{1}{3}\Big(\frac{8}{3} + \ln\frac{v}{4}\Big)v^3 - \frac{1}{3}\Big(\frac{1}{6} + \ln\frac{v}{4}\Big)v^4 + \frac{1}{48}\Big(\frac{43}{12} - \ln\frac{v}{4}\Big)v^5 - \frac{1}{\sqrt{\pi}}\sum_{r=6}^{\infty}\frac{\Gamma(r-9/2)}{(r-1)!\,(r-3)(r-4)(r-5)}v^r\Bigg], \tag{2.7}$$

where $0 < v < 1$. For $p = 4$, (2.5) reduces to

$$f(v) = K(n,4)v^{n-9}(2\pi\iota)^{-1}\int_{L_1}\frac{\Gamma(t+1)\Gamma(t+3)}{\prod_{j=5}^{7}(t+j)\Gamma(t+15/2)\Gamma(t+13/2)}v^{-t}\,dt.$$

The integrand has simple poles at $t = -1, -2$, poles of order two at $t = -r$, $r = 3, 4, 8, 9, \ldots$, and poles of order three at $t = -r$, $r = 5, 6, 7$. Evaluating residues at these poles and using residue theorem, we get the density for $p = 4$ as

$$\begin{aligned} f(v) = K(n,4)v^{n-9}\Bigg[&\frac{v}{660\,\Gamma^2(\frac{11}{2})} - \frac{v^2}{270\,\Gamma^2(\frac{9}{2})} - \frac{1}{168\,\Gamma^2(\frac{7}{2})}\left(\frac{2521}{420} + \ln\frac{v}{16}\right)v^3 \\ &- \frac{1}{90\,\Gamma^2(\frac{5}{2})}\left(\frac{71}{15} + \ln\frac{v}{16}\right)v^4 + \frac{1}{36\pi}\Big\{\left(\ln\frac{v}{16} + \frac{31}{12}\right)^2 + \frac{1781}{144} - \frac{2\pi^2}{3}\Big\}\frac{v^5}{2} \\ &- \frac{1}{360\pi}\Big\{\left(\frac{127}{60} - \ln\frac{v}{16}\right)^2 + \frac{31769}{3600} - \frac{2\pi^2}{3}\Big\}\frac{v^6}{2} \\ &- \frac{15}{2(6!)^2\pi}\Big\{\Big(\frac{121}{30} - \ln\frac{v}{16}\Big)^2 + \frac{33}{200} - \frac{2\pi^2}{3}\Big\}\frac{v^7}{2} \\ &+ \frac{1}{\pi^2}\sum_{j=8}^{\infty}\Big\{\psi(j) + \psi(j-2) \\ &+ \sum_{i=5}^{7}\frac{1}{j-i} - \psi\left(j - \frac{13}{2}\right) - \psi\left(j - \frac{11}{2}\right) - \ln v\Big\} \\ &\frac{\Gamma(j-11/2)\Gamma(j-13/2)}{(j-1)!(j-3)!\,(j-5)(j-6)(j-7)}v^j\Bigg],\ 0 < v < 1. \end{aligned} \tag{2.8}$$

For $p = 5$, (2.5) slides to

$$f(v) = K(n,5)v^{n-11}(2\pi\,\iota)^{-1}\int_{L_1}\frac{\Gamma(t+1)\Gamma(t+3)}{\prod_{j=5}^{9}(t+j)^{a_j}\Gamma(t+17/2)\Gamma(t+19/2)}v^{-t}\,dt,$$

where $a_5 = a_6 = a_8 = a_9 = 1$ and $a_7 = 2$. The integrand has simple poles at $t = -1, -2$, poles of order 2 at $t = -r$, $r = 3, 4, 10, 11, \ldots$, poles of order 3 at $t = -5, -6, -8, -9$ and a pole of order 4 at $t = -7$. Evaluating residues at these poles and using residue theorem, the density is derived as

$$f(v) = K(n,5)v^{n-11}\Bigg[A_1^{(0)}v + A_2^{(0)}v^2 + (-\ln v + B_3^{(0)})A_3^{(0)}v^3 + (-\ln v + B_4^{(0)})A_4^{(0)}v^4 + \sum_{r=5,6,8,9}\left\{(-\ln v + B_r^{(0)})^2 + B_r^{(1)}\right\}A_r^{(0)}\frac{v^r}{2} + \left\{(-\ln v + B_7^{(0)})^3 + 3(-\ln v)B_7^{(1)} + 3B_7^{(0)}B_7^{(1)} + B_7^{(2)}\right\}A_7^{(0)}\frac{v^7}{3!} + \sum_{r=10}^{\infty}(-\ln v + B_r^{(0)})A_r^{(0)}v^r\Bigg], 0 < v < 1, \tag{2.9}$$

where $A_1^{(0)} = \frac{12}{(10)!\Gamma^2(15/2)}$, $A_2^{(0)} = -\frac{4}{65(7)!\Gamma^2(13/2)}$, $A_3^{(0)} = \frac{1}{44(6)!\Gamma^2(11/2)}$, $B_3^{(0)} = -\frac{5219}{693} + \ln(16)$, $A_4^{(0)} = \frac{2}{27(6)!\Gamma^2(9/2)}$, $B_4^{(0)} = -\frac{1691}{252} + \ln(16)$, $A_5^{(0)} = \frac{5}{(8)!\Gamma^2(7/2)}$, $B_5^{(0)} = -\frac{569}{105} + \ln(16)$, $B_5^{(1)} = \frac{1202849}{88200} - \frac{2\pi^2}{3}$, $A_6^{(0)} = -\frac{1}{15(6)!\Gamma^2(5/2)}$, $B_6^{(0)} = -\frac{69}{20} + \ln(16)$, $B_6^{(1)} = \frac{10969}{720} - \frac{2\pi^2}{3}$, $A_7^{(0)} = \frac{7}{2(9)!\Gamma^2(3/2)}$, $B_7^{(0)} - \frac{2}{15} + \ln(16)$, $B_7^{(1)} = \frac{24947}{1800} - \frac{2\pi^2}{3}$, $B_7^{(2)} = 24\zeta(3) - \frac{1504261}{54000}$, $A_8^{(0)} = \frac{1}{3(5)!(7)!\pi}$, $B_8^{(0)} = \frac{989}{210} + \ln(16)$, $B_8^{(1)} = \frac{911681}{88200} - \frac{2\pi^2}{3}$, $A_9^{(0)} = \frac{1}{4(8)!(6)!(4)!\pi}$, $B_9^{(0)} = \frac{4831}{840} + \ln(16)$, $B_9^{(1)} = \frac{488573}{705600} - \frac{2\pi^2}{3}$, $A_r^{(0)} = -\frac{\Gamma(r-15/2)\Gamma(r-17/2)}{\pi^2(r-1)!(r-3)!\prod_{j=5}^{9}(r-j)^{a_j}}$ and $B_r^{(0)} = \psi(r) + \psi(r-2) + \sum_{j=5}^{9} a_j(r-j)^{-1} - \psi(r - \frac{15}{2}) - \psi(r - \frac{17}{2})$.

3 Computation

The computation of the exact percentage points has been carried out by using the CDF $F(v,p) = \int_0^v f(t)\,dt$ where $f(t)$ is given by (2.6), (2.7), (2.8) and (2.9). The CDF $F(v,p)$ for $p = 2,3,4,5$ is obtained by integrating term by term these density functions. For each p, $F(v,p)$ is computed for various values of v to check the monotonicity and conditions such as $F(v,p) \to 0$ as $v \to 0$ and $F(v,p) \to 1$ as $v \to 1$. Then, v is computed for $p = 2,3,4,5$. These are given in Table 1. We have used MATHEMATICA 12.0 to carry out these computations. To compute v for given value of $\alpha = F(v,p)$, we have used `FindRoot` which searches for a numerical solution to the given equation using Newton's method or a variant of the secant method. A six place accuracy has been kept throughout. Tables are given for $p = 2,3,4,5$.

Acknowledgments: This research was supported by the Comité para el Desarrollo de la Investigación, Universidad de Antioquia research grant no. 2023-66291.

References

[1] T. M. Apostol, Zeta and related functions, *NIST Handbook of Mathematical Functions*. Edited by Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark, U. S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

[2] R. A. Askey, R. Roy, Gamma function, *NIST Handbook of Mathematical Functions*. Edited by Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark, U. S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

[3] Liliam Cardeño, Daya K. Nagar, Testing block sphericity of a covariance matrix, *Divulg. Mat.*, **9**, no. 1, (2001), 25–34.

[4] C. C. Chao, A. K. Gupta, Testing of homogeneity of diagonal blocks with blockwise independence, *Comm. Statist. Theory Methods*, **20,** (1991), no. 7, 1957–1969.

[5] A. K. Gupta, C. C. Chao, Nonnull distribution of the likelihood ratio criterion for testing the equality of diagonal blocks with blockwise independence, *Acta Math. Sci.*, **14,** (1994), 195–203.

[6] Arjun K. Gupta, Daya K. Nagar, Armando Gómez, Percentage points of LRC for testing equality of covariance matrices under intraclass correlation structure, *Comm. Stat. Simulation Comput.*, **46,** no. 5, (2017), 3855–3869.

[7] D. K. Nagar, A. K. Gupta, Percentage points for testing homogeneity of several univariate Gaussian populations, *Appl. Math. Comput.*, **156,** (2004), no. 2, 551–561.

[8] Daya K. Nagar, Edwin Zarrazola, Percentage points for testing homogeneity of covariance matrices of several bivariate Gaussian populations, *Int. J. Math. Comput. Sci.*, **16,** no. 2, (2021), 597–605.

[9] I. Olkin, Testing and estimation for structures which are circular symmetric in blocks, *Multivariate Statistical Inference*, (D. G. Kabe, R. P. Gupta, eds.), North-Holland, (1973), 183–195.

[10] D. Roland Thomas, Univariate repeated measures techniques applied to multivariate data, *Psychometrika*, **48,** no. 3, (1983), 451–464.

[11] Edwin Zarrazola, Raúl Alejandro Morán-Vásquez, Daya K. Nagar, Percentage points for testing two-sample compound symmetry, *Int. J. Math. Comput. Sci.*, **18,** no. 3, (2023), 571–580.

Percentage points of V for $p = 2$

$n\backslash\alpha$	0.01	0.025	0.05	0.1
4	0.0027	0.0067	0.0136	0.0278
5	0.0350	0.0565	0.0820	0.1205
6	0.0907	0.1265	0.1641	0.2152
7	0.1520	0.1966	0.2406	0.2973
8	0.2111	0.2607	0.3077	0.3661
9	0.2654	0.3174	0.3655	0.4238
10	0.3144	0.3673	0.4153	0.4724
11	0.3582	0.4111	0.4583	0.5138
12	0.3973	0.4497	0.4958	0.5494
13	0.4324	0.4838	0.5286	0.5802
14	0.4638	0.5141	0.5575	0.6072
15	0.4922	0.5411	0.5832	0.6310
16	0.5178	0.5655	0.6061	0.6521
17	0.5411	0.5874	0.6267	0.6710
18	0.5623	0.6073	0.6453	0.6879
19	0.5817	0.6253	0.6621	0.7032
20	0.5995	0.6419	0.6775	0.7171
21	0.6158	0.6570	0.6915	0.7297
22	0.6309	0.6709	0.7043	0.7413
23	0.6449	0.6838	0.7162	0.7519
24	0.6579	0.6957	0.7271	0.7617
25	0.6700	0.7068	0.7372	0.7707
26	0.6812	0.7170	0.7466	0.7791
27	0.6918	0.7266	0.7554	0.7869
28	0.7016	0.7356	0.7636	0.7942
29	0.7109	0.7440	0.7712	0.8010
30	0.7196	0.7519	0.7784	0.8074

Percentage points of V for $p = 3$

$n\backslash\alpha$	0.01	0.025	0.05	0.1
6	0.0009	0.0023	0.0046	0.0095
7	0.0135	0.0221	0.0325	0.0488
8	0.0394	0.0559	0.0738	0.0993
9	0.0726	0.0958	0.1196	0.1515
10	0.1090	0.1373	0.1653	0.2016
11	0.1460	0.1782	0.2091	0.2482
12	0.1823	0.2172	0.2501	0.2909
13	0.2172	0.2540	0.2881	0.3299
14	0.2502	0.2883	0.3231	0.3653
15	0.2813	0.3202	0.3553	0.3975
16	0.3104	0.3497	0.3850	0.4269
17	0.3377	0.3772	0.4123	0.4537
18	0.3632	0.4026	0.4374	0.4783
19	0.3870	0.4262	0.4606	0.5009
20	0.4093	0.4481	0.4821	0.5216
21	0.4301	0.4686	0.5020	0.5408
22	0.4497	0.4876	0.5206	0.5585
23	0.4680	0.5054	0.5378	0.5750
24	0.4852	0.5221	0.5539	0.5903
25	0.5014	0.5377	0.5689	0.6046
26	0.5166	0.5524	0.5830	0.6179
27	0.5310	0.5662	0.5962	0.6304
28	0.5446	0.5792	0.6086	0.6421
29	0.5574	0.5914	0.6203	0.6531
30	0.5696	0.6030	0.6314	0.6634

Percentage points of V for $p = 4$

$n\backslash\alpha$	0.01	0.025	0.05	0.1
8	0.0003	0.0008	0.0017	0.0035
9	0.0054	0.0089	0.0132	0.0201
10	0.0171	0.0247	0.0331	0.0453
11	0.0342	0.0458	0.0580	0.0749
12	0.0548	0.0702	0.0857	0.1066
13	0.0776	0.0963	0.1147	0.1387
14	0.1017	0.1231	0.1438	0.1703
15	0.1262	0.1499	0.1725	0.2010
16	0.1507	0.1763	0.2004	0.2303
17	0.1749	0.2020	0.2272	0.2583
18	0.1985	0.2268	0.2529	0.2848
19	0.2214	0.2506	0.2774	0.3098
20	0.2434	0.2734	0.3006	0.3334
21	0.2647	0.2952	0.3227	0.3557
22	0.2850	0.3160	0.3437	0.3767
23	0.3045	0.3357	0.3635	0.3965
24	0.3232	0.3545	0.3824	0.4153
25	0.3411	0.3725	0.4002	0.4330
26	0.3581	0.3895	0.4172	0.4497
27	0.3744	0.4058	0.4333	0.4655
28	0.3900	0.4213	0.4486	0.4805
29	0.4050	0.4360	0.4632	0.4947
30	0.4192	0.4501	0.4770	0.5082

Percentage points of V for $p = 5$

$n\backslash\alpha$	0.01	0.025	0.05	0.1
10	0.0001	0.0003	0.0006	0.0013
11	0.0022	0.0036	0.0054	0.0084
12	0.0074	0.0108	0.0147	0.0204
13	0.0159	0.0215	0.0276	0.0363
14	0.0269	0.0350	0.0433	0.0547
15	0.0401	0.0505	0.0609	0.0748
16	0.0549	0.0674	0.0798	0.0959
17	0.0709	0.0853	0.0994	0.1175
18	0.0876	0.1038	0.1194	0.1392
19	0.1048	0.1226	0.1395	0.1608
20	0.1222	0.1414	0.1594	0.1819
21	0.1397	0.1601	0.1791	0.2026
22	0.1570	0.1785	0.1983	0.2227
23	0.1742	0.1965	0.2171	0.2421
24	0.1911	0.2142	0.2353	0.2609
25	0.2077	0.2314	0.2530	0.2791
26	0.2238	0.2481	0.2701	0.2965
27	0.2396	0.2643	0.2866	0.3133
28	0.2549	0.2801	0.3026	0.3295
29	0.2699	0.2953	0.3180	0.3450
30	0.2843	0.3100	0.3328	0.3600

Table 1: percentage points of V for $p = 2, 3, 4$ and 5.