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On the Diophantine equation $63^x + 323^y = z^2$

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Abstract

In this article, we prove that $(1, 0, 8)$ and $(0, 1, 18)$ are the only solutions for the Diophantine equation $63^x + 323^y = z^2$ where x, y and z are non-negative integers.

1 Introduction

The Catalan's conjecture [1], proved by Mihailescu [2], plays an important role in finding the non-negative solutions to the Diophantine equations of the form $a^x + b^y = z^2$, where a and b are fixed. In 2014, Sroysang [3] proved that the Diophantine equation $323^x + 325^y = z^2$ has a unique non-negative integer solution and that it is $(x, y, z) = (1, 0, 18)$. In the same year, Sroysang [4] showed that $(0, 1, 8)$ is the only one non-negative integer solution (x, y, z) of the Diophantine equation $5^x + 63^y = z^2$.

In this paper, we solve the Diophantine equation $63^x + 323^y = z^2$, where x, y and z are non-negative integers.

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2 Preliminaries

Throughout this paper, $a \equiv_m b$ always means that a is congruent to b modulo m where a, b, and m are integers such that $m \geqslant 1$. Moreover, we denote $a \equiv_m b$ or $a \equiv_m c$ by $a \equiv_m b, c$.

Next, we recall two lemmas [3] and [4] which will be useful in the prove of our main theorem.

Lemma 2.1. [3] *The Diophantine equation* $1 + 323^y = z^2$ *has the unique non-negative integer solution* $(y, z) = (1, 18)$.

Lemma 2.2. [4] *The Diophantine equation* $63^x + 1 = z^2$ *has the unique non-negative integer solution* $(x, z) = (1, 8)$.

3 Main Results

Before considering the main result, we shall give two lemmas.

Lemma 3.1. *If* z *is an integer, then* $z^2 \equiv_{17} 0, 1, 2, 4, 8, 9, 13, 15, 16.$

Proof. Assume that z is an integer. Then

$$z \equiv_{17} 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16.$$

Thus,

$$z^2 \equiv_{17} 0, 1, 4, 9, 16, 8, 2, 15, 13, 13, 15, 2, 8, 16, 9, 4, 1.$$

The proof of this lemma is complete. □

Lemma 3.2. *If* n *is a positive odd integer, then* $12^{2n-1} \equiv_{17} 3, 5, 6, 7, 10, 11, 12, 14.$

Proof. We will prove by induction that $12^{2n-1} \equiv_{17} 3, 5, 6, 7, 10, 11, 12, 14$, for all $n \in \mathbb{N}$. If $n = 1$, we have $12^1 \equiv_{17} 12$. Thus, the statement is true for $n = 1$. Assume that it is true for $n = k$. Then, $12^{2k-1} \equiv_{17} 3, 5, 6, 7, 10, 11, 12, 14$, and so $12^{2k+1} \equiv_{17} 7, 6, 14, 5, 12, 3, 11, 10$. Hence, the statement is true for $n = k + 1$, which proves the result. □

Next, we give our main result.

Theorem 3.3. *The Diophantine equation* $63^x + 323^y = z^2$ *has exactly two non-negative integer solutions* (x, y, z), *which are* $(1, 0, 8)$ *and* $(0, 1, 18)$.

Proof. Assume that there exist non-negative integers x, y, z such that $63^x + 323^y = z^2$. By Lemma 2.1 and 2.2 , we obtain that $(0, 1, 18)$ and $(1, 0, 8)$ are two solutions (x, y, z) of the equation. Now, we consider $x \geqslant 1$ and $y \geqslant 1$. If y is odd, then $z^2 = 63^x + 323^y \equiv_3 2$, which contradicts the fact that $z^2 \equiv_3 0, 1$. Now, y is even. If x is even, then $z^2 = 63^x + 323^y \equiv_4 2$, which contradicts the fact that $z^2 \equiv_4 0, 1$. Then, x is odd. Since $63 \equiv_{17} 12$, we have $63^x \equiv_{17} 3, 5, 6, 7, 10, 11, 12, 14$, by Lemma 3.2. Since $323^y \equiv_{17} 0$, $z^2 \equiv_{17} 3, 5, 6, 7, 10, 11, 12, 14$, which contradicts Lemma 3.1. This completes the proof. □

Corollary 3.4. $(1, 0, 4)$ *and* $(0, 1, 9)$ *are the only non-negative integer solutions* (x, y, w) *of the Diophantine equation* $63^x + 323^y = 4w^2$.

4 Conclusion

In this paper, we showed that the Diophantine equation $63^x + 323^y = z^2$ has only the two solutions $(1, 0, 8)$ and $(1, 0, 18)$, when x, y, z are non-negative integers.

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