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Exploring $8^x + n^y = z^2$ through Associated Elliptic Curves

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Abstract

This paper investigates the exponential Diophantine equation $8^x + n^y = z^2$, where $n > 1$ is an odd positive integer. We characterize solutions for the base cases ($x = 0$ or $y = 0$) and describe, based on implications of Bennett and Skinner's theorem, that no solutions exist for $y > 2$ in certain cases. For $y = 1$ and $y = 2$, we employ elliptic curve methods, focusing on the equations $z^2 = t^3 + n$ and $z^2 = t^3 + n^2$, where $t = 2^x$. This work generalizes known results for specific cases and provides insights into this class of Diophantine equations and their associated elliptic curves.

1 Introduction

Exponential Diophantine equations of the form $a^x + b^y = z^2$, where a, b are fixed positive integers and x, y, z are non-negative integer variables, have been a subject of intense study in number theory. Recent research has focused on specific cases, particularly $8^x + p^y = z^2$ where p is prime.

Sroysang [2, 4] and Rabago [3] investigated cases where $p = 19, 13$, and 17, respectively. In a more general context, Qi and Li [5] examined $8^x + p^y = z^2$

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for any odd prime p, classifying solutions based on the congruence class of p modulo 8. Manikandan and Venkatraman [6] and Panraksa [8] studied the equation $8^x + 161^y = z^2$, providing a complete characterization of its solutions.

We investigate the equation $8^x + n^y = z^2$ for positive integers n, with a computational focus on $n = pq$ where p and q are distinct odd primes. Our examples and detailed computations are for prime pairs (p, q) with $p, q < 200$, combining theoretical analysis with practical computational results.

2 Preliminaries

Our analysis relies heavily on a theorem by Bennett and Skinner [1], which provides powerful results for a class of ternary Diophantine equations. We present a simplified version of their theorem that is directly applicable to our problem:

Theorem 2.1. *(Bennett and Skinner, 2004) Let D be an odd positive integer. Then:*
(a). The equation $x^2 + D^m = 2^n$ has no solutions in integers (x, m, n) with $m > 1$, unless $(|x|, m, n, D) = (13, 3, 9, 7)$.
(b). The equation $x^2 - D^m = 2^n$, with $D > 1$, $m > 2$ and $n > 1$, has only the integer solution $(|x|, m, n, D) = (71, 3, 7, 17)$.

This Theorem 2.1 (b) will be crucial in our analysis of the equation $8^x + n^y = z^2$, particularly for the case where $y > 2$. It provides strong restrictions on the possible solutions, which we will exploit in our proofs.

For the cases where $y = 1$ and $y = 2$, we will employ results from the theory and computation of elliptic curves.

3 Base Cases

We begin by examining the base cases ($x = 0$ or $y = 0$) of the equation $8^x + n^y = z^2$, where n is a positive integer.

Proposition 3.1. *For the exponential Diophantine equation $8^x + n^y = z^2$, where $n > 1$ is a positive integer, if $xy = 0$, then:*
1. No solutions exist when $x = y = 0$.
2. For $x = 0$ and $y > 0$:
Subcase 1. When $y = 1$, solutions exist if and only if n has one of the

following two forms:
(a) $n = (2k)^2 - 1$ *for some positive integer* k*, in which case the unique solution is* $(x, y, z) = (0, 1, 2k)$*.*
(b) $n = (2k+1)^2 - 1$ *for some non-negative integer* k*, in which case the unique solution is* $(x, y, z) = (0, 1, 2k+1)$*.*
Subcase 2. No solutions exist for $y > 1$*.*

3. $(x, y, z) = (1, 0, 3)$ *is the only solution when* $x > 0$ *and* $y = 0$*.*

Proof. 1. When $x = y = 0$: The equation becomes $1 + 1 = z^2$, which has no integer solutions.

2. When $x = 0$ and $y > 0$: We have $n^y = z^2 - 1$.

For $y > 1$, the right-hand side cannot be a perfect power greater than 1 and so there are no solutions.

For $y = 1$, we consider two cases:

Case 1: n is odd. In this case, z must be even. Let $z = 2k$, where $k \geq 1$. Then

$$n = (2k)^2 - 1 = 4k^2 - 1$$

This yields the solution $(x, y, z) = (0, 1, 2k)$, where $n = (2k-1)(2k+1)$.

Case 2: n is even. In this case, z must be odd. Let $z = 2k + 1$, where $k \geq 0$. Then

$$n = ((2k+1) - 1)((2k+1) + 1) = (2k)(2k+2) = 4k^2 + 4k = (2k+1)^2 - 1$$

This yields the solution $(x, y, z) = (0, 1, 2k+1)$, where $n = (2k+1)^2 - 1$.

3. When $x > 0$ and $y = 0$, we have the equation $8^x + 1 = z^2$. For $x = 1$, we obtain the solution $(1, 0, 3)$. For $x > 1$, we can rewrite the equation as $8^x = (z-1)(z+1)$. Note that z must be odd and at least 7 (since $x > 1$). Let $z = 2k + 1$ where $k \geq 3$. Substituting this into our equation gives $8^x = 4k(k+1)$. This implies that $k(k+1)$ is divisible only by a power of 2. However, since k and $k+1$ are consecutive integers with $k \geq 3$, one of them must be odd and greater than 3. This leads to a contradiction for $x > 1$, as $k(k+1)$ would have an odd factor greater than 3. □

4 Elliptic Curve Analysis and Computational Method

For this section, we assume that $n > 1$ is an odd positive integer and y a positive integer.

Bennett and Skinner's work [1] allows us to focus our study of $8^x + n^y = z^2$ on $y = 1$ and $y = 2$. We transform the equation into elliptic curves. For $y = 1$, the equation becomes $z^2 = t^3 + n$, and for $y = 2$, it becomes $z^2 = t^3 + n^2$, where $t = 2^x$ in both cases. Both equations are in Weierstrass form $y^2 = x^3 + ax + b$, with $x = t$, $y = z$, $a = 0$, and $b = n$ or n^2, respectively.

Using SageMath [7], we implement a two-step process to find solutions. First, we find integral points on these elliptic curves. Then, we identify the points where t is a power of 2. This approach allows for a systematic search for solutions for any given n, providing a comprehensive method to analyze the original Diophantine equation.

To find solutions to the equation $8^x + n^y = z^2$ for $y = 1$ and $y = 2$, we implemented a computational method using SageMath. The core of this method is the find_solutions function:

```
def is_power_of_two(n):
    if not n.is_integer():
        return False
    n = Integer(n)
    return n > 0 and (n & (n - 1)) == 0

def find_solutions(n):
    solutions = []
    for y in [1, 2]:
        E = EllipticCurve([0, 0, 0, 0, n^y])
        for point in E.integral_points():
            t, z, _ = point
            if is_power_of_two(t):
                x = Integer(t).nbits() - 1
                solutions.append((x, y, abs(z)))
    return solutions
```

Figure 1: SageMath Code for Finding Solutions to $8^x + n^y = z^2$.

This function constructs elliptic curves $z^2 = t^3 + n^y$ for $y = 1$ and $y = 2$, finds their integral points, and identifies solutions where t is a power of 2.

Example 4.1. *To demonstrate both the general method and the direct elliptic curve calculations, let us consider* $n = 161 = (7)(23)$*, corresponding to the equation* $8^x + 161^y = z^2$ *studied by Manikandan and Venkatraman [6] and Panraksa [8].*

First, we compute the integral points on the elliptic curves directly:

```
E1 = EllipticCurve([0, 0, 0, 0, 161])
print(E1.integral_points())

E2 = EllipticCurve([0, 0, 0, 0, 161^2])
print(E2.integral_points())


n = Integer(161)
print(find_solutions(n))
```

Output:

```
[(-5 : -6 : 1), (2 : -13 : 1), (4 : -15 : 1),
(190 : -2619 : 1)]
[(-28 : -63 : 1), (0 : -161 : 1), (92 : -897 : 1)]
```

These are all the integral points on the curves $z^2 = t^3 + 161$ *and* $z^2 = t^3 + 161^2$*, respectively.*

Now, let us use our function to identify which of these points correspond to solutions of our original equation: Output:

```
[(1, 1, 13), (2, 1, 15)]
```

These solutions can be verified:

$(1, 1, 13)$*:* $8^1 + 161^1 = 8 + 161 = 169 = 13^2$.
$(2, 1, 15)$*:* $8^2 + 161^1 = 64 + 161 = 225 = 15^2$.

Comparing the outputs, we can see that our function correctly identified the points (2 : 13 : 1) and (4 : 15 : 1) from the first curve as solutions, corresponding to $t = 2 = 2^1$ *and* $t = 4 = 2^2$*, respectively. It ignored the points with negative* t *values and the point (190 : 2619 : 1) where* t *is not a power of 2. For the second curve* ($y = 2$)*, no solutions were found as none of the* t *values are powers of 2.*

We extended this analysis to various prime pairs (p, q) where $n = pq$. Table 1 presents the additional solutions (beyond the trivial solution $(x, y, z) = (1, 0, 3)$) for prime pairs (p, q) with $p < q < 200$.

Table 1: Additional Solutions to $8^x + (pq)^y = z^2$ for Odd Prime Pairs (p, q) with $p < q < 200$

(p, q)	(x, y, z)	(p, q)	(x, y, z)
(3, 5)	(0, 1, 4), (2, 2, 17)	(31, 191)	(1, 1, 77)
(3, 19)	(2, 1, 11)	(37, 53)	(2, 1, 45)
(3, 43)	(4, 1, 65)	(41, 43)	(0, 1, 42)
(3, 131)	(4, 1, 67)	(41, 97)	(3, 1, 67)
(5, 7)	(0, 1, 6)	(41, 137)	(1, 1, 75)
(7, 23)	(1, 1, 13), (2, 1, 15)	(41, 193)	(1, 1, 89)
(7, 31)	(1, 1, 15), (3, 1, 27)	(43, 59)	(2, 1, 51)
(7, 47)	(3, 1, 29)	(47, 79)	(1, 1, 61), (3, 1, 65), (5, 1, 191)
(7, 103)	(1, 1, 27), (5, 1, 183)	(53, 181)	(4, 1, 117)
(7, 167)	(3, 1, 41)	(59, 61)	(0, 1, 60)
(7, 191)	(3, 1, 43)	(67, 83)	(2, 1, 75)
(11, 13)	(0, 1, 12)	(71, 73)	(0, 1, 72)
(11, 139)	(4, 1, 75)	(71, 127)	(1, 1, 95)
(13, 29)	(2, 1, 21)	(71, 199)	(3, 1, 121), (4, 1, 135)
(17, 19)	(0, 1, 18)	(73, 89)	(2, 1, 81)
(17, 89)	(1, 1, 39), (3, 1, 45)	(73, 113)	(7, 1, 1451)
(23, 31)	(3, 1, 35)	(79, 103)	(3, 1, 93)
(23, 47)	(1, 1, 33)	(89, 193)	(3, 1, 133)
(23, 151)	(1, 1, 59), (4, 1, 87)	(97, 113)	(2, 1, 105)
(29, 31)	(0, 1, 30)	(101, 103)	(0, 1, 102)
(29, 157)	(4, 1, 93)	(103, 191)	(5, 1, 229)
(31, 47)	(2, 1, 39), (5, 1, 185)	(107, 109)	(0, 1, 108)
(31, 71)	(1, 1, 47), (5, 1, 187)	(127, 199)	(1, 1, 159)
(31, 79)	(7, 1, 1449)	(137, 139)	(0, 1, 138)
(149, 151)	(0, 1, 150)	(179, 181)	(0, 1, 180)
(151, 167)	(2, 1, 159)	(181, 197)	(2, 1, 189)
(157, 173)	(2, 1, 165)	(191, 193)	(0, 1, 192)
(163, 179)	(2, 1, 171)	(197, 199)	(0, 1, 198)

This computational method, combining direct elliptic curve calculations with our specialized function, allows for efficient exploration of solutions across various values of n, providing a powerful tool for studying the properties and patterns of the equation $8^x + n^y = z^2$.

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