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Rarely (τ_1, τ_2)-continuous functions

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Abstract

In this paper, we introduce the notion of rarely (τ_1, τ_2)-continuous functions. Some characterizations of rarely (τ_1, τ_2)-continuous functions are also investigated.

1 Introduction

In 1979, Popa [10] introduced and investigated an important concept of rare continuity as a generalization of weak continuity due to Levine [7]. This concept has been further studied by Long and Herrington [8] and Jafari [6]. Jafari [5] also generalized the concept of rare continuity to rare β-continuity by involving the notion of β-open sets. Caldas [3] introduced a new class of

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functions called rarely $\beta\theta$-continuous functions by utilizing the notion of β-θ-open sets and investigated some characterizations of rarely $\beta\theta$-continuous functions. Jafari [4] introduced and studied the concept of rare α-continuity as a generalization of rare continuity and weak α-continuity [9]. In this paper, we introduce the notion of rarely (τ_1, τ_2)-continuous functions. We also investigate some characterizations of rarely (τ_1, τ_2)-continuous functions.

2 Preliminaries

Throughout this paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [2] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [2] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [2] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [11] if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$. A subset R of a bitopological space (X, τ_1, τ_2) is called a $\tau_1\tau_2$-*rare set* if $\tau_1\tau_2\text{-Int}(R) = \emptyset$.

Lemma 2.1. *Let (X, τ_1, τ_2) be a bitopological space. Then, $\tau_1\tau_2$-Int$(F \cup R) \subseteq F$ for every $\tau_1\tau_2$-rare set R and every $\tau_1\tau_2$-closed set F.*

3 Rarely (τ_1, τ_2)-continuous functions

We begin this section by introducing the notion of rarely (τ_1, τ_2)-continuous functions.

Definition 3.1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be rarely (τ_1, τ_2)-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ and a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V \cup R_V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called rarely (τ_1, τ_2)-continuous if f has this property at each point of X.*

Theorem 3.2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is rarely (τ_1, τ_2)-continuous at $x \in X$;

(2) for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \tau_1\tau_2\text{-}Int(f^{-1}(V \cup R_V))$;

(3) for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2\text{-}Cl(V) \cap R_V = \emptyset$ such that

$$x \in \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V) \cup R_V));$$

(4) for each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \tau_1\tau_2\text{-}Int(f^{-1}(V \cup R_V))$;

(5) for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2\text{-}Int(f(U) \cap (Y - V)) = \emptyset$;

(6) for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2\text{-}Int(f(U)) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (1), there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ and a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V \cup R_V$. Thus, $x \in U \subseteq f^{-1}(V \cup R_V)$ and hence $x \in \tau_1\tau_2\text{-Int}(f^{-1}(V \cup R_V))$.

(2) $\Rightarrow$ (3): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ such that

$$x \in \tau_1\tau_2\text{-Int}(f^{-1}(V \cup R_V)).$$

Let $R'_V = R_V \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))$. Then, we have $R'_V \cap \sigma_1\sigma_2\text{-Cl}(V) = \emptyset$ and R'_V is a $\sigma_1\sigma_2$-rare set. Since

$$\begin{aligned}\sigma_1\sigma_2\text{-Cl}(V) \cup R'_V &= \sigma_1\sigma_2\text{-Cl}(V) \cup [R_V \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))] \\ &= \sigma_1\sigma_2\text{-Cl}(V) \cup R_V \supseteq V \cup R_V.\end{aligned}$$

Thus, $x \in \tau_1\tau_2\text{-Int}(f^{-1}(V \cup R_V)) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V) \cup R'_V))$.

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. By (3), there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2\text{-Cl}(V) \cap R_V = \emptyset$ such that

$$x \in \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V) \cup R_V)).$$

Let $R''_V = R_V \cup (\sigma_1\sigma_2\text{-Cl}(V) - V)$. By Lemma 2.1, R''_V is a $\sigma_1\sigma_2$-rare set and $R''_V \cap V = \emptyset$. Thus, $x \in \tau_1\tau_2\text{-Int}(f^{-1}(V \cup R''_V))$.

(4) $\Rightarrow$ (5): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, $f(x) \in V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-open. By (4), there exists a $\sigma_1\sigma_2$-rare set R_V with

$$\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cap R_V = \emptyset$$

and $x \in \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cup R_V))$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cup R_V)$. Thus, $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cup R_V$ and by Lemma 2.1, we have

$$\begin{aligned}\sigma_1\sigma_2\text{-Int}(f(U) \cap (Y - V)) &= \sigma_1\sigma_2\text{-Int}(f(U)) \cap \sigma_1\sigma_2\text{-Int}(Y - V)\\ &\subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V) \cup R_V) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))\\ &\subseteq (\sigma_1\sigma_2\text{-Cl}(V) \cup \sigma_1\sigma_2\text{-Int}(R_V)) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))\\ &= \sigma_1\sigma_2\text{-Cl}(V) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset\end{aligned}$$

and hence $\sigma_1\sigma_2\text{-Int}(f(U) \cap (Y - V)) = \emptyset$.

(5) $\Rightarrow$ (6): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then by (5), there exists a $\tau_1\tau_2$-open set U of X containing x such that

$$\sigma_1\sigma_2\text{-Int}(f(U) \cap (Y - V)) = \emptyset.$$

Thus, $\sigma_1\sigma_2\text{-Int}(f(U)) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset$ and hence $\sigma_1\sigma_2\text{-Int}(f(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$.

(6) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (6), there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2\text{-Int}(f(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Let $M_V = f(U) \cap (Y - V)$. Then, we have

$$\begin{aligned}\sigma_1\sigma_2\text{-Int}(M_V) &\subseteq \sigma_1\sigma_2\text{-Int}(f(U)) \cap \sigma_1\sigma_2\text{-Int}(Y - V)\\ &= \sigma_1\sigma_2\text{-Int}(f(U)) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset.\end{aligned}$$

Therefore, M_V is a $\sigma_1\sigma_2$-rare set and $M_V \cap V = \emptyset$. Let $N_V = \sigma_1\sigma_2\text{-Cl}(V) - V$. Then, N_V is a $\sigma_1\sigma_2$-closed $\sigma_1\sigma_2$-rare set such that $N_V \cap V = \emptyset$. Thus, $R_V = M_V \cup N_V$ is a $\sigma_1\sigma_2$-rare set and $R_V \cap V = \emptyset$. By Lemma 2.1, $\sigma_1\sigma_2\text{-Int}(R_V) = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Int}(R_V)) = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Int}(M_V \cup N_V)) \subseteq \sigma_1\sigma_2\text{-Int}(N_V) = \emptyset$. Therefore,

$$\begin{aligned}f(U) &= [f(U) - \sigma_1\sigma_2\text{-Int}(f(U))] \cup \sigma_1\sigma_2\text{-Int}(f(U))\\ &\subseteq [f(U) - \sigma_1\sigma_2\text{-Int}(f(U))] \cup \sigma_1\sigma_2\text{-Cl}(V)\\ &= [(f(U) \cap (V \cup (Y - V))) - \sigma_1\sigma_2\text{-Int}(f(U))] \cup [(\sigma_1\sigma_2\text{-Cl}(V) - V) \cup V]\\ &= [((f(U) \cap V) \cup (f(U) \cap (Y - V))) - \sigma_1\sigma_2\text{-Int}(f(U))] \cup (N_V \cup V)\\ &\subseteq [V \cup (f(U) \cap (Y - V))] \cup (N_V \cup V)\\ &= V \cup (M_V \cup N_V) = V \cup R_V.\end{aligned}$$

Thus, there exists a $\sigma_1\sigma_2$-rare set $R_V = M_V \cup N_V$ such that $R_V \cap V = \emptyset$ and $f(U) \subseteq V \cup R_V$. This shows that f is rarely (τ_1, τ_2)-continuous at $x \in X$. $\square$

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