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Almost nearly (τ_1, τ_2)-continuous multifunctions and the $\tau_1\tau_2$-δ-closure operator

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Abstract

In this paper, we investigate some characterizations of upper and lower almost nearly (τ_1, τ_2)-continuous multifunctions by utilizing the $\tau_1\tau_2$-δ-closure operator.

1 Introduction

In 2003, Ekici [5] introduced and studied the concept of nearly continuous multifunctions. Moreover, Ekici [4] introduced and investigated the notion of

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almost nearly continuous multifunctions as a generalization of nearly continuous multifunctions and almost continuous multifunctions [7]. Noiri and Popa [6] introduced and studied the notion of almost nearly m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space. Chutiman et al. [3] defined upper almost nearly (τ_1, τ_2)-continuous multifunctions and lower almost nearly (τ_1, τ_2)-continuous multifunctions. In this paper, we investigate several characterizations of upper almost nearly (τ_1, τ_2)-continuous multifunctions and lower almost nearly (τ_1, τ_2)-continuous multifunctions by utilizing the $\tau_1\tau_2$-δ-closure operator.

2 Preliminaries

Throughout the paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [2] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [2] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [2] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [9] if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$. The complement of a $(\tau_1, \tau_2)r$-open set is called $(\tau_1, \tau_2)r$-*closed*. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-δ-*open* [1] if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$-δ-*closed* [1]. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$-δ-*interior* [1] of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$-δ-*closure* [1] of A and is denoted by $\tau_1\tau_2$-δ-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\mathscr{N}(\tau_1, \tau_2)$-*closed* [8] if every cover of A by $(\tau_1, \tau_2)r$-open sets of X has a finite subcover.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X \mid F(x) \subseteq B\}$$

and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$.

3 Characterizations of upper and lower almost nearly (τ_1, τ_2)-continuous multifunctions

In this section, we investigate some characterizations of upper almost nearly (τ_1, τ_2)-continuous multifunctions and lower almost nearly (τ_1, τ_2)-continuous multifunctions.

Definition 3.1. [3] *A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be upper almost nearly* (τ_1, τ_2)*-continuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $F(x)$ *and having* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed complement, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $F(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$*. A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be upper almost nearly* (τ_1, τ_2)*-continuous if* F *has this property at every point of* X*.*

Lemma 3.2. [3] *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper almost nearly* (τ_1, τ_2)*-continuous;*

(2) $F^+(V) \subseteq \tau_1\tau_2\text{-}Int(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *having* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed complement;*

(3) $\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^-(K)$ *for every* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed and* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ *for every every subset* B *of* Y *having the* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed* $\sigma_1\sigma_2$*-closure;*

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ *for every every subset* B *of* Y *such that* $Y - \sigma_1\sigma_2\text{-}Int(B)$ *is* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed;*

(6) $F^+(V)$ *is* $\tau_1\tau_2$*-open in* X *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y *having* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed complement;*

(7) $F^-(K)$ *is* $\tau_1\tau_2$*-closed in* X *for every* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed and* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Theorem 3.3. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper almost nearly* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2$-*Cl*($F^-(\sigma_1\sigma_2$-*Cl*($\sigma_1\sigma_2$-*Int*($\sigma_1\sigma_2$-δ-*Cl*(B))))) $\subseteq F^-(\sigma_1\sigma_2$-$\delta$-*Cl*($B$)) *for every subset* B *of* Y *having the* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed* $\sigma_1\sigma_2$-δ*-closure;*

(3) $\tau_1\tau_2$-*Cl*($F^-(\sigma_1\sigma_2$-*Cl*($\sigma_1\sigma_2$-*Int*($\sigma_1\sigma_2$-*Cl*(B))))) $\subseteq F^-(\sigma_1\sigma_2$-$\delta$-*Cl*($B$)) *for every subset* B *of* Y *having the* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed* $\sigma_1\sigma_2$-δ*-closure.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y such that $\sigma_1\sigma_2$-δ-Cl(B) is $\mathscr{N}(\sigma_1, \sigma_2)$-closed. By Lemma 3.2 of [1], $\sigma_1\sigma_2$-δ-Cl(B) is $\sigma_1\sigma_2$-closed. Then, $\sigma_1\sigma_2$-δ-Cl(B) is $\sigma_1\sigma_2$-closed and $\mathscr{N}(\sigma_1, \sigma_2)$-closed. Thus by Lemma 3.2,

$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B)).$$

(2) $\Rightarrow$ (3): This is obvious since $\sigma_1\sigma_2$-Cl(B) $\subseteq \sigma_1\sigma_2$-$\delta$-Cl($B$) for every subset B of Y.

(3) $\Rightarrow$ (1): Let K be any $\mathscr{N}(\sigma_1, \sigma_2)$-closed and $(\sigma_1, \sigma_2)r$-closed set of Y. By (3), we have $\tau_1\tau_2$-Cl($F^-(K)$) $= \tau_1\tau_2$-Cl($F^-(\sigma_1\sigma_2$-Cl($\sigma_1\sigma_2$-Int(K)))) $=$ $\tau_1\tau_2$-Cl($F^-(\sigma_1\sigma_2$-Cl($\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(K))))) $\subseteq F^-(\sigma_1\sigma_2$-$\delta$-Cl($K$)) $= F^-(K)$ and hence $F^-(K)$ is $\tau_1\tau_2$-closed in X. By Lemma 3.2, F is upper almost nearly (τ_1, τ_2)-continuous. ☐

Definition 3.4. [3] *A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be lower almost nearly* (τ_1, τ_2)*-continuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$ *and having* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed complement, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $\sigma_1\sigma_2$-*Int*($\sigma_1\sigma_2$-*Cl*(V)) $\cap F(z) \neq \emptyset$ *for each* $z \in U$. *A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be lower almost nearly* (τ_1, τ_2)*-continuous if* F *has this property at every point of* X.

Theorem 3.5. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower almost nearly* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2$-*Cl*($F^+(\sigma_1\sigma_2$-*Cl*($\sigma_1\sigma_2$-*Int*($\sigma_1\sigma_2$-δ-*Cl*(B))))) $\subseteq F^+(\sigma_1\sigma_2$-$\delta$-*Cl*($B$)) *for every subset* B *of* Y *having the* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed* $\sigma_1\sigma_2$-δ*-closure;*

(3) $\tau_1\tau_2$-*Cl*($F^+(\sigma_1\sigma_2$-*Cl*($\sigma_1\sigma_2$-*Int*($\sigma_1\sigma_2$-*Cl*(B))))) $\subseteq F^+(\sigma_1\sigma_2$-$\delta$-*Cl*($B$)) *for every subset* B *of* Y *having the* $\mathscr{N}(\sigma_1, \sigma_2)$*-closed* $\sigma_1\sigma_2$-δ*-closure.*

Proof. The proof is similar to that of Theorem 3.3. ☐

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