International Journal of Mathematics and
Computer Science, **20**(2025), no. 1, 49–52
DOI: https://doi.org/10.69793/ijmcs/01.2025/bayah

Independence-Preserving Operations: Effects in Polynomial Representations

Benjier H. Arriola[1], **Shaleema A. Arriola**[1],
Bayah J. Amiruddin-Rajik[2], **Sherlyn U. Sappayani**[2]

[1]Mathematics Department
College of Education
Basilan State College, Sumagdang, Isabela City
7300 Basilan, Philippines

[2]Mindanao State University
Tawi-Tawi College of Technology and Oceanography
7500 Bongao, Tawi-Tawi, Philippines

email: benj_arriola@yahoo.com, shallarriola1023@yahoo.com,
bayahamiruddin@msutawi-tawi.edu.ph,
sherlynsappayani@msutawi-tawi.edu.ph

(Received May 24, 2024, Accepted July 3, 2024, Published July 10, 2024)

Abstract

In this piece of work, we show that independent sets are preserved in the join operation of graphs. The adjacency property of the join of graphs guarantees a nice representation of the independent neighborhood polynomial of the join of graphs in terms of the independent neighborhood polynomials of graphs being considered.

1 Introduction

The study of polynomials as graph representation captured a lot of attention recently because of the applications of these representations in other fields of science such as Chemistry, Biology, and Physics [3]. In 1994, Hoede and

Key words and phrases: Neighborhood system, Independent neighborhood polynomial.
AMS (MOS) Subject Classifications: 05C25, 05C30, 05C31.
ISSN 1814-0432, 2025, http://ijmcs.future-in-tech.net

Li [4] established the independent set polynomial of graphs which counts the number of independent subsets of the vertex-set of a graph. In 2023, Arriesgado, Salim and Artes Jr. [2] investigated the common neighborhood polynomial of the join of graph.

A subset S of $V(G)$ is said to be *independent* in G if the elements of S are pairwise non-adjacent in G. If $v \in V(G)$, the *open neighborhood* or simply the *neighborhood* of v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. For a subset S of $V(G)$, the *neighborhood system* of S in G is the set $N_G(S) = \bigcup_{s \in S} N_G(s) \setminus S$ [2].

The *independent neighborhood polynomial* of a graph G of order n in the indeterminates x and y, denoted by $\Gamma_{in}(G; x, y)$, is given by

$$\Gamma_{in}(G; x, y) = \sum_{j=0}^{n-1} \sum_{i=1}^{\alpha(G)} \alpha_{ij}(G) x^i y^j,$$

where $\alpha_{ij}(G)$ is the number of independent subsets of $V(G)$ of cardinality i with neighborhood system cardinality equal to j and $\alpha(G)$ is the independence number of G. This concept was first introduced by Amiruddin-Rajik et al. [1].

2 Results

The following result characterizes the independent subsets of complete q-partite graphs.

Lemma 2.1. *Consider a natural number $q \geq 3$ and an increasing sequence $\langle r_i \rangle_{i=1}^{q}$ of natural numbers. A subset S of $V(K_{r_1, r_2, \ldots, r_q})$ is independent in $K_{r_1, r_2, \ldots, r_q}$ if and only if $S \subseteq V_i$ for some partite set V_i of $K_{r_1, r_2, \ldots, r_q}$.*

Proof. Note that for each $i \in \{1, 2, \ldots, q\}$, the elements of V_i are mutually non-adjacent, and so does any subset of V_i. If for $i \neq j$, S contains an element of V_i and an element of V_j, then S will not be independent.

Conversely, assume that $S \subseteq V_i$ for some $i \in \{1, 2, \ldots, q\}$. Since V_i is independent, S is also independent in V_i being a subset of an independent set. The adjacency property of $K_{r_1, r_2, \ldots, r_q}$ asserts that S is independent in $K_{r_1, r_2, \ldots, r_q}$. □

Next, we establish the polynomial representation for the neighborhood systems of independent sets in complete q-partite graphs.

Theorem 2.2. *Suppose that $\langle r_i \rangle_{i=1}^{q}$ is an increasing sequence of natural numbers where $q \geq 3$. Then*

$$\Gamma_{in}(K_{r_1,r_2,\ldots,r_q}; x, y) = \sum_{i=1}^{q} [(1+x)^{r_i} - 1] y^{\sum_{j \neq i} r_j}.$$

Proof. Assume that S is a subset of the vertex set of $K_{r_1,r_2,\ldots,r_q}$. Further, assume that S is independent in $K_{r_1,r_2,\ldots,r_q}$. Then by Lemma 2.1, $S \subseteq V_i$ for some $i \in \{1, 2, \ldots, q\}$. The independence of V_i asserts that there are exactly $\binom{r_i}{k}$ independent subsets of V_i for each $1 \leq i \leq q$ and for $1 \leq k \leq r_i$. If $S \subseteq V_i$, then the neighborhood system of S in $K_{r_1,r_2,\ldots,r_q}$ is the set $\bigcup_{j \neq i} V_j = V(K_{r_1,r_2,\ldots,r_q}) \setminus V_i$. The cardinality of this set is $\left| \bigcup_{j \neq i} V_j \right| = \sum_{j \neq i} r_j$. For each independent set $S \subset V_i$, this gives $x^{|S|} y^{\sum_{j \neq i} r_j}$. Thus,

$$\begin{aligned} \Gamma_{in}(K_{r_1,r_2,\ldots,r_q}; x, y) &= \sum_{i=1}^{q} \sum_{k=1}^{r_i} \binom{r_i}{k} x^k y^{\sum_{j \neq i} r_j} \\ &= \sum_{i=1}^{q} \left[\sum_{k=1}^{r_i} \binom{r_i}{k} x^k \right] y^{\sum_{j \neq i} r_j} \\ &= \sum_{i=1}^{q} [(1+x)^{r_i} - 1] y^{\sum_{j \neq i} r_j}. \end{aligned}$$

This completes the proof. □

The next lemma establishes the independence-preserving property of the join of graphs.

Lemma 2.3. *A subset S of $V(G \oplus H)$ is independent in $G \oplus H$ if and only if S is an independent subset of $V(G)$ or S is an independent subset of $V(H)$.*

Proof. Assume that S is independent in $G \oplus H$. Suppose $\{u, v\} \subseteq S$ with $u \in V(G)$ and $v \in V(H)$. Then $uv \in V(G \oplus H)$ by the adjacency property of the join operation of graphs. This implies that either S is contained in $V(G)$ or S is contained in $V(H)$. Hence, either S is an independent subset of $V(G)$ or S is an independent subset of $V(H)$.

Conversely, assume that S is an independent subset of $V(G)$ or S is an independent subset of $V(H)$. Then the adjacency property of $G \oplus H$ asserts that S is independent in $G \oplus H$. □

The next result establishes the polynomial representation of the join of graphs with respect to independent neighborhood systems.

Theorem 2.4. *Let G and H be non-trivial graphs. Then*

$$\Gamma_{in}(G \oplus H; x, y) = \Gamma_{in}(G; x, y)y^{|V(H)|} + \Gamma_{in}(H; x, y)y^{|V(G)|}.$$

Proof. Consider a subset S of the vextex-set of the join $G \oplus H$. Suppose that S be an independent in $G \oplus H$. Now, Lemma 2.3 asserts that either S is an independent set in G or S is an independent set in H. If S is an independent set in G, then $N_{G\oplus H}(S) = N_G(S) \cup V(H)$. This gives the first term. Similar arguement holds when the set S is an independent set in H. This gives the second term. The proof is complete. □

References

[1] B.J. Amiruddin-Rajik, R.A. Sappayani, R.G. Artes Jr., B.I. Junio, H.H. Moh. Jiripa, Independent neighborhood polynomial of a graph, *Advances and Applications in Discrete Mathematics*, **41,** no. 2, (2024), 149–156. http://dx.doi.org/10.17654/0974165824010

[2] A.L. Arriesgado, J.I.C. Salim, R.G. Artes Jr., Clique Connected Common Neighborhood Polynomial of the Join of Graphs, *International Journal of Mathematics and Computer Science*, **18,** no. 4, (2023), 655–659.

[3] J. Ellis-Monaghan, J. Merino, *Graph Polynomials and Their Applications II: Interrelations and Interpretations*, Birkhauser, Boston, 2011.

[4] C. Hoede, X. Li, Clique polynomials and independent set polynomials of graphs, *Discrete Mathematics*, **125,** (1994), 219–228.