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Polynomial Representation of Biclique Neighborhood in Graphs

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Abstract

In this piece of work, we introduce the idea of biclique neighborhood polynomial of graphs. This polynomial provides a representation of graph structure by considering the number of bicliques of specified cardinality with the corresponding neighborhood systems cardinality. Moreover, we generate results on the polynomial representations of biclique neighborhood of some graph structures.

1 Introduction

The study of graph polynomials garnered a lot of attention in recent years because of the applications of these polynomials in other fields of sciences such as Chemistry, Biology, and Physics [3]. In 2022, Artes, Langamin,

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and Calib-og [2] introduced a bivariate graph polynomial called the clique common neighborhood polynomial of a graph by considering the common neighborhood system of a clique in a graph. In 2023, Arriesgado, Salim and Artes Jr. [1] investigated the clique connected common neighborhood polynomial of the join of graphs. Recently, Lumpayao et al. [4] explored a pioneering work on biclique polynomial of a graph.

We introduce a new graph polynomial as follows:
The *biclique neighborhood polynomial* of a graph G is given by $\Gamma_{bn}(G; x, y) = \sum_{i=2}^{b(G)} b_{ij}(G)x^i y^j$, where $b_{ij}(G)$ is the number of bicliques in G of cardinality i with corresponding neighborhood system cardinality equal to j and $b(G)$ is the cardinality of a maximum biclique in G.

2 Results

The following result characterizes the bicliques in complete graphs.

Lemma 2.1. [4] *Suppose n is a natural number such that $n \geq 2$. A subset S of $V(K_n)$ is a biclique in K_n if and only if $|S| = 2$.*

The biclique neighborhhod polynomial of K_n is established below.

Theorem 2.2. *Let $n \geq 2$ be a natural number. Then*

$$\Gamma_{bn}(K_n; x, y) = \binom{n}{2} x^2 y^{n-2}.$$

Proof. Let S be a biclique in K_n. Then, by Lemma 2.1, $|S| = 2$. Let $u \in S$. By the adjacency property of K_n, $uw \in E(K_n)$ for every $w \in V(K_n) \setminus S$. Thus, $N_{K_n}(S) = V(K_n) \setminus S$. Hence, $|N_{K_n}(S)| = |V(K_n) \setminus S| = n - 2$. Note that there are $\binom{n}{2}$ subsets of $V(K_n)$ of cardinality 2. The result follows. ☐

The next lemma characterizes the bicliques in paths.

Lemma 2.3. [4] *Suppose n is a natural number such that $n \geq 2$. A subset S of $V(P_n)$ is a biclique in P_n if and only if S induces a subpath of length 1 or 2.*

The biclique neighborhood of a path structure is established in the following result.

Theorem 2.4. *Let $n \geq 5$ be a natural number. Then*

$$\Gamma_{bn}(P_n; x, y) = 2x^2y + (n-3)x^2y^2 + 2x^3y + (n-4)x^3y^2.$$

Proof. Let S be a biclique in P_n. Lemma 2.3 asserts that either S induces a $K_{1,1}$ or S induces a $K_{1,2}$ in P_n. Let $P_n = [v_1, v_2, \ldots, v_n]$.

Case 1. S induces a $K_{1,1}$ in P_n.

If $S = \{v_1, v_2\}$, then $N_{P_n}(S) = \{v_3\}$. Also, if $S = \{v_{n-1}, v_n\}$, then $N_{P_n}(S) = \{v_{n-2}\}$. Hence, $|N_{P_n}(S)| = 1$. This gives the first term. Now, if $S = \{v_i, v_{i+1}\}$ for $i \in \{2, 3, \ldots, n-2\}$, then $N_{P_n}(S) = \{v_{i-1}, v_{i+2}\}$. Hence, $|N_{P_n}(S)| = 2$. The second term follows.

Case 2. S induces a $K_{1,2}$ in P_n.

If $S = \{v_1, v_2, v_3\}$. then $N_{P_n}(S) = \{v_4\}$. Hence, $|N_{P_n}(S)| = 1$. Also, if $S = \{v_{n-2}, v_{n-1}, v_n\}$, then $N_{P_n}(S) = \{v_{n-3}\}$. Hence, $|N_{P_n}(S)| = 1$. This gives the third term. Now, if $S = \{v_i, v_{i+1}, v_{i+2}\}$, where $i \in \{2, 3, \ldots, n-3\}$, then $N_{P_n}(S) = \{v_{i-1}, v_{i+3}\}$. Hence, $|N_{P_n}(S)| = 2$. Hence, the last term follows. □

The next lemma characterizes the bicliques in complete bipartite graphs.

Lemma 2.5. [4] *A subset S of $V(K_{m,n})$ is a biclique in $K_{m,n}$ if and only if S intersects both the partite sets of $K_{m,n}$.*

Finally, the following result establishes the biclique neighborhood polynomial of complete bipartite graphs.

Theorem 2.6. *Let m, n be natural numbers. Then*

$$\Gamma_{bn}(K_{m,n}; x, y) = (x+y)^{m+n} - (x+y)^m y^n - (x+y)^n y^m + y^{m+n}.$$

Proof. Let S be a biclique in $K_{m,n}$. Then by Lemma 2.5, S intersects both the partite sets of $K_{m,n}$. Let V_1 and V_2 be the corresponding partite sets of $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$. Let $u \in S \cap V_1$ and $v \in S \cap V_2$. Then $uw \in E(K_{m,n})$ for every $w \in V_2$. Hence, $N_{K_{m,n}}(u) = V_2$. Similarly, $vz \in E(K_{m,n})$ for every $z \in V_1$. Thus, $N_{K_{m,n}}(v) = V_1$. Accordingly, $N_{K_{m,n}}(S) = V(K_{m,n}) \setminus S$. Note that there are $\binom{m}{i}$ ways of taking i-subsets from m-set and there are $\binom{n}{j}$ ways of taking j-subsets from n-set. Hence we have,

$$\begin{aligned}
\Gamma_{bn}(G;x,y) &= \sum_{i=1}^{m}\sum_{j=1}^{n}\binom{m}{i}\binom{n}{j}x^{i+j}y^{(m+n)-(i+j)} \\
&= \left[\sum_{i=1}^{m}\binom{m}{i}x^{i}y^{m-i}\right]\left[\sum_{j=1}^{n}\binom{n}{j}x^{j}y^{n-j}\right] \\
&= [(x+y)^m - y^m][(x+y)^n - y^n] \\
&= (x+y)^{m+n} - (x+y)^m y^n - (x+y)^n y^m + y^{m+n}.
\end{aligned}$$

This completes the proof. □

References

[1] A.L. Arriesgado, J.I.C. Salim, R.G. Artes Jr., Clique Connected Common Neighborhood Polynomial of the Join of Graphs, *International Journal of Mathematics and Computer Science*, **18,** no. 4, (2023), 655–659.

[2] R.G. Artes Jr., M.A. Langamin, A.B. Calib-og., Clique Common Neighborhood Polynomial of Graphs, *Advances and Applications in Discrete Mathematics*, **35**, (2022), 77–85.

[3] J. Ellis-Monaghan, J. Merino, *Graph Polynomials and Their Applications II: Interrelations and Interpretations*. Birkhauser, Boston, 2011.

[4] S.M. Lumpayao, R.A. Rasid, R.G. Artes Jr., On Biclique Polynomials, *Advances and Applications in Discrete Mathematics*, **41,** no. 3, (2024), 231–237.