International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 1, (2025), 345–351
DOI: https://doi.org/10.69793/ijmcs/01.2025/AhmedHussain

Shape Preserving Multi Approximation by Comonotone Functions

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(Received June 15, 2024, Revised September 27, 2024,
Accepted October 14, 2024, Published November 18, 2024)

Abstract

We prove a multi direct comonotone theorem for the comonotone approximation of piecewise monotone functions via multi polynomials.

1 Introduction

Approximating a piecewise monotone function by comonotone polynomials has been a topic of intense study in recent years. Using comonotone polynomials of degree less than or equal to n to approximate such a function f, lliev [6] and Newman [8] showed that this isn't worse than $Cw\left(f, \frac{1}{n}\right)(f, 1/n)$, where C is an absolute constant dependent only on the quantity of changes

Key words and phrases: Piecewise monotone functions, Comonotone approximation.
AMS (MOS) Subject Classifications: 41A10, 41A63, 41A65, 46E30.
ISSN 1814-0432, 2025, https://future-in-tech.net

in monotonicity of f. For functions that are only continuous, this rate of approximation is known as the Jackson rate.

In this paper, we demonstrate that the error in the comonotone approximation fulfills the appropriate higher order Jackson estimate when the derivative of the piecewise monotone function is continuous. We prove the following theorem.

Theorem 1.1. *Suppose that g is a function that has continuous differentiability in $[-1,1]^\Gamma$ and monotonicity changes ℓ times, $1 \leq \ell < \infty$, in $[-1,1]^\Gamma$. Then $\forall m \geq 1$, $\exists$ a multi polynomial v_m of degree less or equal to m that is comonotone with the function $g(y)$ on $[-1,1]^\Gamma$ and has the property that*

$$\|g - V_m\| \leq \frac{K}{m^\Gamma}\omega(g', \Omega), \quad \Omega = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right). \tag{1.1}$$

where the constant $K(\ell, \Gamma)$ depends exclusively on ℓ and Γ.

Here, $\|\cdot\|$ is the usual sup norm of g and $y = (y_1, \ldots, y_\Gamma)$.

In 1977, DeVore [3] demonstrated that monotone polynomials of degree less or equal to n with error $0\left(n^{-r}\omega\left(f^{(r)}, \frac{1}{n}\right)\right)$ can approximate a monotone function with r continuous derivatives. DeVore also provided a similar conclusion for splines (for an alternative proof of DeVore's spline result, see Beatson [1]). Leviatan and Mhaskar [7] have found such an estimate for the comonotone approximation by splines. Moreover, many researchers have also worked on this topic [2, 4, 5].

In what follows, Π_m refers to the collection of multi polynomials with degree less or equal to m. In addition, $K_1, K_2, \ldots$ stand for constants that are independent of g, m, and ℓ.

2 Proof of the basic result

The flipped function $\tilde{g}$, which undergoes one less change in monotonicity, is evaluated in relation to the error in the comonotone multi-approximation. The following lemma predicts the error in comonotone multi approximation of g.

Lemma 2.1. *There is a constant K_1 that has the following characteristic: Let $g \in C^1[-1,1]^\Gamma$ be a piecewise monotone multivariate function, with one* *change in monotonicity occurring at zero, where g has a zero, and all other changes of the form $\ell \geq 1$. Define*

$$\tilde{g}\left(y_1, \ldots, y_\Gamma\right) = \begin{cases} g\left(y_1, \ldots, y_\Gamma\right), & y_s \geq 0 \\ -g\left(y_1, \ldots, y_\Gamma\right), & y_s < 0 \end{cases}, \quad s = 1, \ldots, \Gamma,$$

and assume that a multi polynomial $V_m \in \Pi_m$ comonotone with $\tilde{g}$ exists for some $m \geq 1$ and $\epsilon > \omega\left(\tilde{g}', \Omega\right)$ such that

$$\|\tilde{g} - V_m\| \leq \epsilon / m^\Gamma, \quad \|\tilde{g}' - V_m'\| \leq \epsilon. \tag{2.2}$$

If so, a multi polynomial $V_{2m} \in \Pi_{2m}$ comonotone exists whose g satisfies the conditions

$$\|g - V_{2m}\| \leq K_1 \epsilon / m^\Gamma, \quad \|g' - V_{2m}'\| \leq K_1 \epsilon. \tag{2.3}$$

Proof. In comparison to $g, \tilde{g}$ has one fewer monotonicity change. $\forall 0 < |y_s| < h_s/m, h_s \geq 1$,

$$\left|\tilde{g}'\left(y_1, \ldots, y_\Gamma\right)\right| \leq \omega\left(\tilde{g}', \left(\left|y_1\right|, \ldots, \left|y_\Gamma\right|\right)\right) \leq \left(h_1 \ldots h_\Gamma\right) \omega\left(\tilde{g}', \Omega\right). \tag{2.4}$$

Since $\tilde{g}(0, \ldots, 0) = 0$,

$$\left|\tilde{g}\left(y_1, \ldots, y_\Gamma\right)\right| = \left|y_1\right| \ldots \left|y_\Gamma\right| \left|\tilde{g}'\left(y_1, \ldots, y_\Gamma\right)\right| \leq \frac{\left(h_1^2 \ldots h_\Gamma^2\right)}{m^\Gamma} \omega(\tilde{g}, \Omega). \tag{2.5}$$

Following DeVore [3], we create the approximation to $\operatorname{sgn}\left(y_1, \ldots, y_\Gamma\right)$ for any $m > 1$,

$$W_m\left(y_1, \ldots, y_\Gamma\right) = w_m\left(y_1\right) + \cdots + w_m\left(y_\Gamma\right) \ni w_m\left(y_s\right) = K_m \int_0^{y_s} \left(H_n\left(t_s\right) / t_s\right)^4 dt_s,$$

n is the largest odd integer to ensure $W_m \in \Pi_m$ and K_m are selected to ensure $w_m(1) = 1$ such that $W_m(1, \ldots, 1) = w_m(1) + \cdots + w_m(1) = 1 + \cdots + 1 = \Gamma$. W_m is monotone increasing, odd and has the property that

$$\begin{aligned} &\left|\operatorname{sgn}\left(y_1, \ldots, y_\Gamma\right) - W_m\left(y_1, \ldots, y_\Gamma\right)\right| \\ &\quad = \left|\operatorname{sgn}\left(y_1\right) - w_m\left(y_1\right) + \cdots + \operatorname{sgn}\left(y_\Gamma\right) - w_m\left(y_\Gamma\right)\right| \\ &\quad \leq \left|\operatorname{sgn}\left(y_1\right) - w_m\left(y_1\right)\right| + \cdots + \left|\operatorname{sgn}\left(y_\Gamma\right) - w_m\left(y_\Gamma\right)\right| \\ &\quad \leq K_2 \left(\left|m y_1\right|^{-3} + \cdots + \left|m y_\Gamma\right|^{-3}\right) = K_2 \sum_{s=1}^{\Gamma} \left|m y_s\right|^{-3}, \; y \in [-1, 0)^\Gamma \cup (0, 1]^\Gamma \end{aligned}$$

$$\left|\operatorname{sgn}\left(y_1\right) - w_m\left(y_1\right)\right| + \cdots + \left|\operatorname{sgn}\left(y_\Gamma\right) - w_m\left(y_\Gamma\right)\right| \leq 1 + \cdots + 1 = \Gamma, \;\; y \in [-1, 1]^\Gamma. \tag{2.6}$$

Given $\tilde{g}(0,\ldots,0)=0$, we can assume that $V_m(0,\ldots,0)=0$ and replace ϵ in the first inequality of (2.2). Define

$$V_{2m}=\int_0^{y_1}\ldots\int_0^{y_\Gamma}V_m'\left(t_1,\ldots,t_\Gamma\right)W_m\left(t_1,\ldots,t_\Gamma\right)dt_1\ldots dt_\Gamma.$$

Then V_{2m} is comonotone with g, and

$$\begin{aligned}
&g\left(y_1,\ldots,y_\Gamma\right)-V_{2m}\left(y_1,\ldots,y_\Gamma\right)\\
&=\int_0^{y_1}\ldots\int_0^{y_\Gamma}\left[\tilde{g}'\left(t_1,\ldots,t_\Gamma\right)-V_m'\left(t_1,\ldots,t_\Gamma\right)\right]\operatorname{sgn}\left(t_1,\ldots,t_\Gamma\right)dt_1\ldots dt_\Gamma\\
&\quad+\int_0^{y_1}\ldots\int_0^{y_\Gamma}V_m'\left(t_1,\ldots,t_\Gamma\right)\left[\operatorname{sgn}\left(t_1,\ldots,t_\Gamma\right)-W_m\left(t_1,\ldots,t_\Gamma\right)\right]dt_1\ldots dt_\Gamma\\
&=\left(\tilde{g}\left(y_1,\ldots,y_\Gamma\right)-W_m\left(y_1,\ldots,y_\Gamma\right)\right)\operatorname{sgn}\left(y_1,\ldots,y_\Gamma\right)\\
&\quad+\int_0^{y_1}\ldots\int_0^{y_\Gamma}V_m'\left(t_1,\ldots,t_\Gamma\right)\left[\operatorname{sgn}\left(t_1,\ldots,t_\Gamma\right)-W_m\left(t_1,\ldots,t_\Gamma\right)\right]dt_1\ldots dt_\Gamma.
\end{aligned}\tag{2.7}$$

Let $\zeta=(\zeta_1,\ldots,\zeta_\Gamma)\ni\zeta_s=\operatorname{sgn}\left(y_s/m\right), s=1,\ldots,\Gamma$. If $0<|y_s|\le i/m$, $i\ge 1$ then by (2.2), (2.4), (2.6) and $\tilde{g}'(0,\ldots,0)=0$ we get

$$\begin{aligned}
&\left|\int_0^{y_1}\ldots\int_0^{y_\Gamma}V_m'\left(t_1,\ldots,t_\Gamma\right)\left[\operatorname{sgn}\left(t_1,\ldots,t_\Gamma\right)-W_m\left(t_1,\ldots,t_\Gamma\right)\right]dt_1\ldots dt_\Gamma\right|\\
&\le\sum_{\substack{h_s=0\\ s=1,\ldots,\Gamma}}^{i-1}\left|\int_{h_1\zeta_1}^{(h_1+1)\zeta_1}\ldots\int_{h_\Gamma\zeta_\Gamma}^{(h_\Gamma+1)\zeta_\Gamma}V_m'\left(t_1,\ldots,t_\Gamma\right)\left[\operatorname{sgn}\left(t_1,\ldots,t_\Gamma\right)\right.\right.\\
&\qquad\left.\left.-W_m\left(t_1,\ldots,t_\Gamma\right)\right]dt_1\ldots dt_\Gamma\right|\\
&\le\frac{1}{m^\Gamma}\left[\omega\left(\tilde{g}',\Omega\right)+\epsilon\right]+\frac{1}{m^\Gamma}\sum_{\substack{h_h=1\\ s=1,\ldots,\Gamma}}^{i-1}\left[\left(h_1+1\right)\ldots\left(h_\Gamma+1\right)\omega\left(\tilde{g}',\Omega\right)+\epsilon\right]\\
&\qquad\times K_3\left(h_1^{-3}\ldots h_\Gamma^{-3}\right)\\
&\le K_4\frac{\epsilon}{m^\Gamma}.
\end{aligned}$$

The first inequality in (2.3) is obtained by using (2.2) to estimate the other term in (2.7).

In this context, we won't provide the analogous proof for the second inequality in equation (2.3). □

proof of theorem 1.1. When m is small, specifically $m < M(\ell)$, the theorem becomes trivial because $g'(\eta_1, \dots, \eta_\Gamma) = 0$. Consequently,

$$\begin{aligned}&|g(y_1, \dots, y_\Gamma) - g(\eta_1, \dots, \eta_\Gamma)| \\ &\quad \le |(y_1 - \eta_1) \dots (y_\Gamma - \eta_\Gamma)|\, |g'(\alpha_1, \dots, \alpha_\Gamma)| \le 2^\Gamma \omega(g', (2, \dots, 2)) \\ &\quad \le K(\ell, \Gamma) \frac{1}{m^\Gamma} \omega(g', \Omega).\end{aligned}$$

We establish the theorem for large values of m by induction using the number of monotonicity changes as ℓ.

We look at the claim that constants K(ℓ) and M(ℓ) exist, for any multivariate function that is piecewise monotone $g \in C^1[-1,1]^\Gamma$ with $\ell \ge 0$ modifies monotonicity, which in $[-1,1]^\Gamma g'$ has zeros (this assumption is only required when $\ell = 0$), and $\forall m \ge M(\ell)$, there is a $V_m \in \Pi_m$ comonotone with g satisfying

$$\|g - V_m\| \le K(\ell, \Gamma) \frac{1}{m^\Gamma} \omega(g', \Omega), \quad \|g' - V'_m\| \le K(\ell, \Gamma) \frac{1}{m^\Gamma} \omega(g', \Omega).$$

When $\ell = 0$, the claim is true. Without loss of generlity, assume that g vanishes at one of the zeros of g' and then extend g via linear functions to $[-3,3]^\Gamma$ while maintaining the modulus of continuity of g'

Observe that $\max\left\{\|g\|_{[-3,3]^\Gamma,\|}g'\|_{[-3,3]^\Gamma}\right\} \le K_5 m^\Gamma \omega(g', \Omega)$ and define

$$\begin{aligned}&\mho_m(g, (y_1, \dots, y_\Gamma)) \\ &\quad = \int_{-2}^{2} \dots \int_{-2}^{2} \lambda_m((y_1 - t_1), \dots, (y_\Gamma - t_\Gamma))\, g(t_1, \dots, t_\Gamma)\, dt_1 \dots dt_\Gamma + a_m,\end{aligned}$$

where $\{\lambda_m\}$ is an appropriate sequence of kernels for positive multi polynomials and

$$\int_{-4}^{4} \dots \int_{-4}^{4} \lambda_m(t_1, \dots, t_\Gamma) = \Gamma, \int_{-4}^{4} \dots \int_{-4}^{4} \lambda_m(t_1, \dots, t_\Gamma)\, t_1^2 \dots t_\Gamma^2 = 0\left(m^{-2d}\right),$$

and

$$\|\lambda_m\|_{[-4,4]^\Gamma/[-1,1]^\Gamma} = 0\left(m^{-2d}\right).$$

So

$$\begin{aligned}&\mho'_m(g, (y_1, \dots, y_\Gamma)) \\ &\quad = \int_{-2}^{2} \dots \int_{-2}^{2} \lambda_m((y_1 - t_1), \dots, (y_\Gamma - t_\Gamma))\, g'(t_1, \dots, t_\Gamma)\, dt_1 \dots dt_\Gamma \\ &\qquad + \lambda_m((y_1 + 2), \dots, (y_\Gamma + 2))\, g(-2, \dots, -2) \\ &\qquad - \lambda_m((y_1 - 2), \dots, (y_\Gamma - 2))\, g(2, \dots, 2) + a_m.\end{aligned}$$

Where

$$\Big|\lambda_m\left((y_1+2),\ldots,(y_\Gamma+2)\right)g(-2,\ldots,-2)$$
$$-\lambda_m\left((y_1-2),\ldots,(y_\Gamma-2)\right)g(2,\ldots,2)\Big|$$
$$\leq K_6 m^{-\Gamma}\omega\left(g',\Omega\right), \quad y\in[-1,1]^\Gamma.$$

Providing a_m is suitably selected from $\pm K_6 m^{-\Gamma}\omega\left(g',\Omega\right)$, $\mho_m(g)$, it is obvious that it will include the necessary approximation features and has the same constant monotonicity as g. We will demonstrate the proposition's validity for s by supposing that it is true for $s-1$. Extend the definition of g to include $[-3,3]^\Gamma$ if it has monotonicity with $\ell\geq 1$ changes. $g'(\eta)=0$ for some $\eta=(\eta_1,\ldots,\eta_\Gamma)\in(-1,1)^\Gamma$ because g has at least one turning point.

We can see that changing $z=(z_1,\ldots,z_\Gamma)=\frac{1}{2}\left((y_1,\ldots,y_\Gamma)-(\eta_1,\ldots,\eta_\Gamma)\right)$ results in a function $f\left(z_1,\ldots,z_\Gamma\right)=g\left(y_1,\ldots,y_\Gamma\right)$ defined for $y\in[-1,1]^\Gamma$ with a turning point at zero and $\omega\left(f',\Omega\right)\leq 4^\Gamma\omega\left(g',\Omega\right)$ when working with $J\subseteq[-3,3]^\Gamma$ of length 4^Γ and centered at η.

It should be noted that $\omega\left(\tilde{f}',\Omega\right)\leq 2^\Gamma\omega\left(f',\Omega\right)$ and $\tilde{f}$ possesses monotonicity with $\ell-1$ changes. A sequence $\{\delta_m\}_{m=2M(\ell-1)}^{\infty}$ of comonotone approximations to f can be found by the lemma and the inductive hypothesis. The statement for ℓ is proved true by inverting the sequence $\{V_m\left(y_1,\ldots,y_\Gamma\right)\}$, $V_m\left(y_1,\ldots,y_\Gamma\right)=\delta_m\left(z_1,\ldots,z_\Gamma\right)$. □

3 Conclusion

We derived the Jackson type theorem for the comonotone approximation of piecewise monotone functions using multi polynomials.

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