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Simulation Analysis of Asymptotic Normality of Maximum Likelihood Estimation Based on the Fisher Scoring Algorithm in Generalized Poisson Regression Modeling

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Abstract

Maximum Likelihood Estimation (MLE) is a method widely used in statistics to estimate the parameters of statistical models. One of the critical aspects of MLE is its asymptotic properties, especially its asymptotic normality. Asymptotic normality refers to the property where the MLE distribution approaches a normal distribution as the sample size tends to infinity. Meanwhile, Fisher Scoring Algorithm (FSA) stands out as a versatile and efficient tool for parameter estimation in various models and statistical disciplines. This research was conducted to obtain the distribution of the MLE estimator based on the FSA. The simulation study used the Bootstrap Method in the Generalized Poisson Regression Model with 600, 700, 800, 900, and 1000 repetitions. The simulation results show that the MLE estimator based on the FSA has the same distribution as the classical MLE estimator; namely, a Normal distribution.

Key words and phrases: Asymptotic Normality, Maximum Likelihood Estimation, Fisher Scoring Algorithm, Generalized Poisson Regression.
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1 Introduction

Estimating parameters in a statistical model is a fundamental statistical technique known as Maximum Likelihood Estimation (MLE) which provides consistent and efficient estimates, making it a cornerstone of statistical theory [1]. The MLE's existence in high-dimensional logistic regression models has been rigorously established, indicating a sharp 'phase transition' in its behavior [2]. This property is crucial in understanding the behavior of MLE in complex models. The appeal of MLE lies in its asymptotic properties, where, under regular conditions, it is consistent, efficient, and normal [3]. MLE is preferred in classical statistics due to its broad applicability and desirable statistical properties [4]. Several references in the literature focus on the asymptotic normality of MLE. Moreover, Tang and Wang [5] discussed the asymptotic normality and consistency of local maximum likelihood type estimators under certain conditions. This further underscores the significance of asymptotic normality in ensuring the reliability of parameter estimates derived from MLE. The equation of the log-likelihood function of the maximized MLE function is nonlinear. So, in this study, the Fisher Scoring Algorithm (FSA) is used to optimize the MLE parameters in parameter estimation. In machine learning and computational biology, Li and Xu [6] demonstrated the effectiveness of the Fisher score algorithm in feature selection for identifying hub genes in hepatocellular carcinoma. Purba et al. [7] also applied MLE based on Newton-Raphson, Fisher Scoring, and Expectation Maximization Algorithms to accident data based on age categories. These three algorithms provide the same estimation results. However, the FSA can reach convergent conditions with fewer iterations than the other two algorithms. Thus, in this paper, we conduct a simulation study to identify the asymptotic normality of MLE based on the FSA in Generalized Poisson Regression (GPR) Modeling using the Bootstrap Method. GPR models have gained significant attention in statistical analysis due to their flexibility in modeling count data affected by various predictor variables. Meanwhile, the bootstrap method is valuable for conducting simulation studies related to MLE of normality.

2 Generalized Poisson Regression (GPR)

The Generalized Poisson Distribution is used to describe a set of data in accordance with the following probability function:

$$p\left(y_i;\mathbf{\Omega_i},\vartheta\right)=\left(\frac{\Omega_i}{\vartheta\Omega_i+1}\right)^{y_i}\frac{(\vartheta y_i+1)^{y_i-1}}{y_i!}e^{-\left(\frac{\Omega_i(\vartheta y_i+1)}{\vartheta\Omega_i+1}\right)}, i=1,2,3,...,n, \tag{2.1}$$

where $\Omega_i=e^{x\beta}=e^{\beta_0+\sum\limits_{j=1}\beta_j x_{ji}}$ [8].

3 Maximum Likelihood Estimation (MLE)

MLE is a statistical technique that is both reliable and effective for parameter estimation [9]. The log-likelihood function is a critical element of the MLE procedure. The log-likelihood function of GPR is featured in this article.

$$\iota(\boldsymbol{\beta},\vartheta)=\sum_{i=1}^{n}\left(\left(\beta_0+\sum_{j=1}^{p}\beta_j x_{ji}\right)y_i-\ln\left(\vartheta\exp\left(\beta_0+\sum_{j=1}^{p}\beta_j x_{ji}\right)+1\right)y_i+(y_i-1)\right.$$
$$\left.\ln(\vartheta y_i+1)-\frac{(\vartheta y_i+1)}{\vartheta\exp\left(\beta_0+\sum_{j=1}^{p}\beta_j x_{ji}\right)+1}\exp\left(\beta_0+\sum_{j=1}^{p}\beta_j x_{ji}\right)-\ln(y_i!)\right). \tag{3.2}$$

Next, parameter estimators are determined using the Fisher Scoring algorithm.

4 Fisher Scoring Algorithm (FSA)

Determining GPR parameters using the FSA requires a gradient vector and a Fisher information matrix, which is the negative of the expectation value of the second derivative matrix of the log-likelihood function. The gradient vector is given as follows:

$$D(\hat{\gamma}^{(r)})_{(p+2)\times 1}=\begin{bmatrix}\frac{\partial l(\boldsymbol{\beta},\vartheta)}{\partial\beta_0} & \frac{\partial l(\boldsymbol{\beta},\vartheta)}{\partial\beta_j} & \cdots & \frac{\partial l(\boldsymbol{\beta},\vartheta)}{\partial\vartheta}\end{bmatrix}, j=1,2,\ldots,p. \tag{4.3}$$

Meanwhile, the Fisher information matrix used is $I(\hat{\gamma})=-E\left[H\left(\hat{\gamma}^{(r)}\right)\right]$, where $H\left(\hat{\gamma}^{(r)}\right)$ is a Hessian matrix containing the second derivative of the log-likelihood function from the GPR model. Determining the estimated parameter values starts from $r=0$ until it converges in the r-th iteration using the equation $\hat{\gamma}^{(r+1)}=\hat{\gamma}^{(r)}+\left[I(\hat{\gamma}^{(r)})\right]^{-1}D(\hat{\gamma}^{(r)})$. The iteration stops at a convergent state, namely at time $\|\hat{\gamma}^{(r+1)}-\hat{\gamma}^{(r)}\|\le\varepsilon$, where $\varepsilon\ge 0$ and ε have minimal values [10].

5 Numerical Study

The bootstrap method is employed in the simulation study conducted in this section. The technique involves the practice of resampling (repeated sampling) from the original data sample. The advantage of the Bootstrap

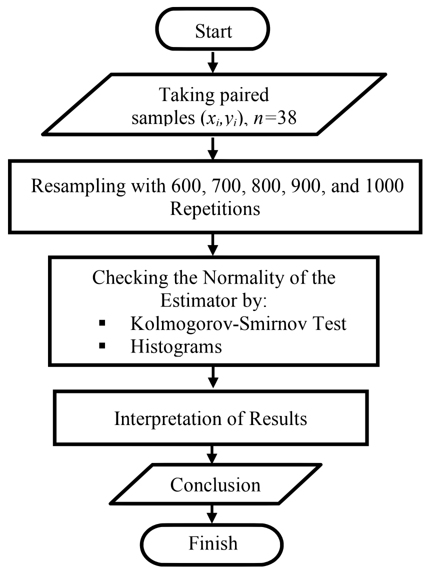

Figure 1: The Research Procedure for the Simulation Study

Method is that this method does not require theoretical model assumptions or mathematical models, and the data does not have to come from a specific distribution [11]. Paired sampling of the response and predictor variables is carried out with a return of $n = 38$. Resampling was carried out 600, 700, 800, 900, and 1000 for observations on 6 estimators produced from the GPR model. Normality checks were carried out using the Kolmogorov-Smirnov and Histogram tests. The flowchart in Figure 1 illustrates the research procedure for a simulation study that employs the Bootstrap Method to ascertain the distribution of the MLE estimator using the Fisher Scoring Algorithm.

6 Results and Discussion

Table 1 provides information that the test decision results for all estimators are the same, namely accepting H0 for all resampling results. Based on Table 2, it is seen from $|D| < D_{(\alpha;n)}$ value for each resampling. The Kolmogorov-Smirnov test has shown that the MLE estimator based on the Fisher-Scoring Algorithm has a Normal distribution. Next, the normality check is demonstrated through a histogram.

Figures 2, 3, 4, 5 and 6 show the histogram plot results for resampling 600, 700, 800, 900, and 1000. The histogram of all resamplings shows that all estimators form a curve that resembles an inverted bell. It is a characteristic

Table 1: Shows The Results of Checking The Normality of The Estimator Using The Kolmogorov-Smirnov Test.

Estimator	**Test Statistics Values** ($\|D\|$)				
	600	**700**	**800**	**900**	**1000**
$\hat{\beta}_0$	0.023162	0.020166	0.023043	0.017072	0.016634
$\hat{\beta}_1$	0.051733	0.050218	0.016694	0.042919	0.026048
$\hat{\beta}_2$	0.042624	0.049959	0.035636	0.038845	0.041964
$\hat{\beta}_3$	0.052826	0.043786	0.040314	0.027899	0.042039
$\hat{\beta}_4$	0.051701	0.029582	0.021627	0.023955	0.040899
$\hat{\beta}_5$	0.032666	0.0177	0.033838	0.027769	0.016914

Table 2: $D_{(\alpha;n)}$ value from The Kolmogorov-Smirnov Table.

	600	**700**	**800**	**900**	**1000**
$D_{(\alpha;n)}$ value	0.05552	0.05140	0.04808	0.04533	0.04301

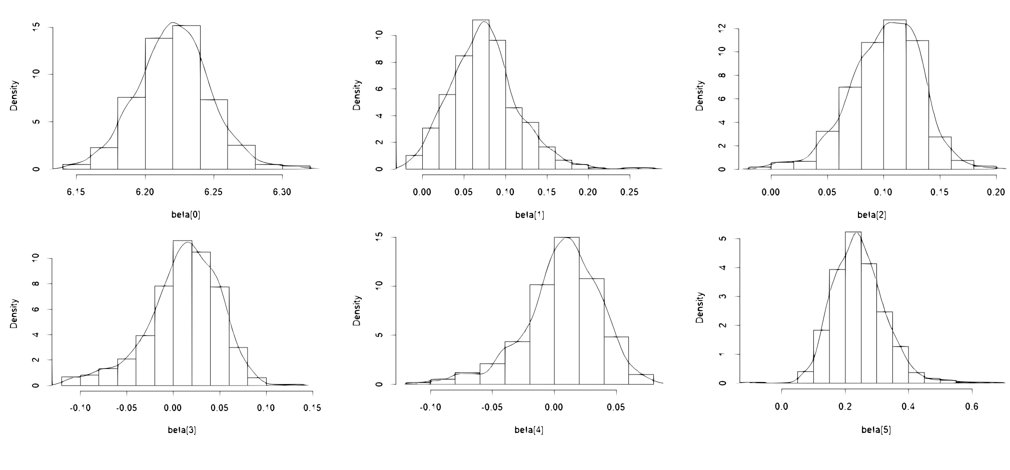

Figure 2: Histogram of parameter estimates for 600 resamplings.

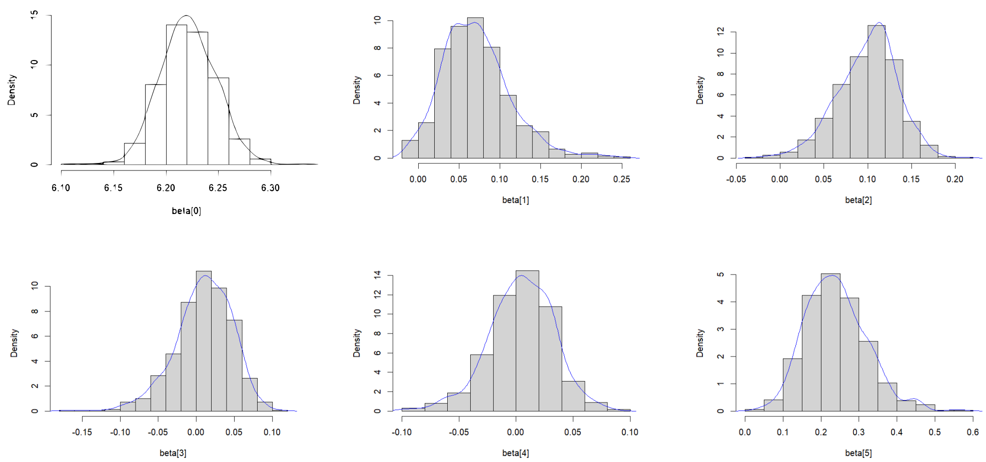

Figure 3: Histogram of parameter estimates for 700 resamplings.

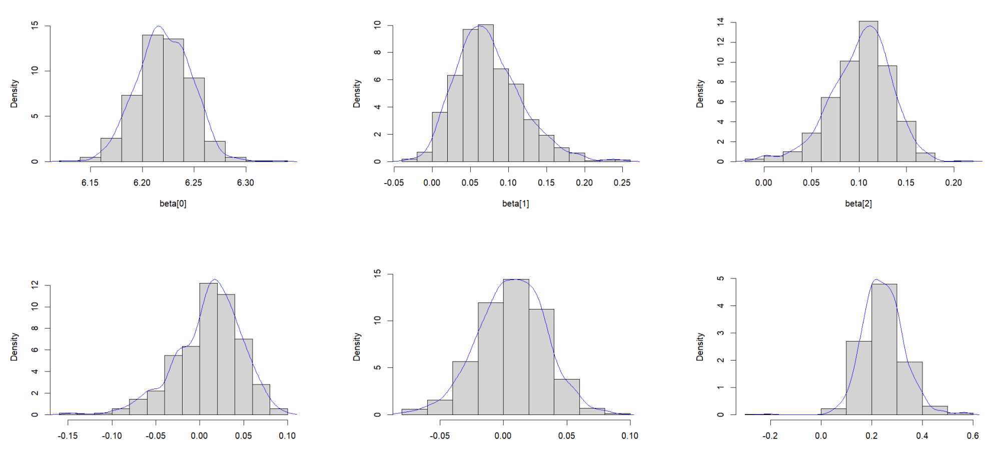

Figure 4: Histogram of parameter estimates for 800 resamplings.

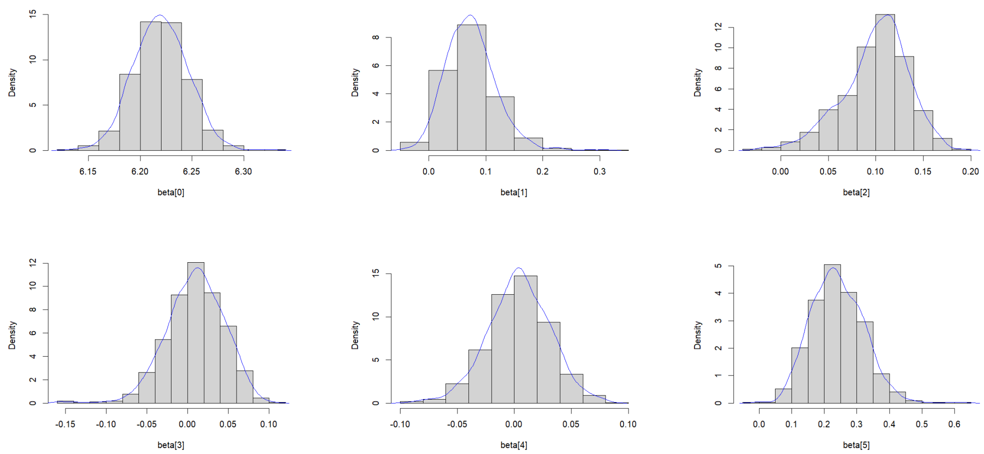

Figure 5: Histogram of parameter estimates for 900 resamplings.

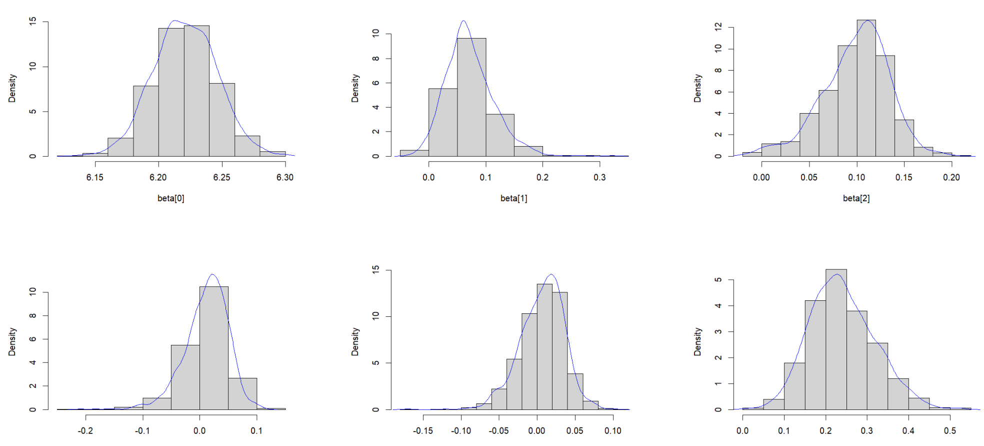

Figure 6: Histogram of parameter estimates for 1000 resamplings.

of a normal distribution. This condition also strengthens the evidence that all estimators produced from all resamplings follow a normal distribution.

7 Conclusion

The simulation study was carried out using resampling of 600, 700, 800, 900, and 1000 in the Generalized Poisson Regression Model. The Kolmogorov-Smirnov test and histogram results conclude that the MLE estimator based on the Fisher-Scoring Algorithm meets the normality assumption. The contribution of this study conveys that further parameter significance testing can use test statistics that have a distribution derived from the Normal distribution.

References

[1] A. Anastasiou, Assessing the multivariate normal approximation of the maximum likelihood estimator from high-dimensional, heterogeneous data, Electronic Journal of Statistics, **12**, no. 2, (2018). https://doi.org/10.1214/18-ejs1492.

[2] E. Candés, P. Sur, The phase transition for the existence of the maximum likelihood estimate in high-dimensional logistic regression, The Annals of Statistics, **48**, no. 1, (2020). https://doi.org/10.1214/18-aos1789.

[3] Y. Jang, J. Zhao, H. Kim, K. Yu, S. Kwon, S. Kim, New closed-form efficient estimator for multivariate gamma distribution, Statistica Neerlandica, **77**, no. 4, (2023), 555–572. https://doi.org/10.1111/stan.12299

[4] R. N. I. Dinnullah, S. Abusini, R. Fitriani, Marjono, Application of Generalized Poisson Regression for the development of modelling motorcycle accidents based on rider characteristics, Journal of Physics: Conference Series, 2634, no. 1, (2023), 012007. https://doi.org/10.1088/1742-6596/2634/1/012007.

[5] M. Tang, Y. Wang, Variable bandwidth local maximum likelihood type estimation for diffusion processes, Advances in Difference Equations, 2018, no. 1, (2018). https://doi.org/10.1186/s13662-018-1506-1.

[6] C. Li, J. Xu, Feature selection with the fisher score followed by the maximal clique centrality algorithm can accurately identify the hub genes of hepatocellular carcinoma, Scientific Reports, **9**, no. 1, (2019). https://doi.org/10.1038/s41598-019-53471-0.

[7] S. A. Purba, Sutarman, S. Darnius, Maximum Likelihood based on Newton Raphson, Fisher Scoring, and Expectation Maximization Algorithm application on accident data, International Journal of Advanced Researh, **6**, no. 1, (2018), 965–969. http://dx.doi.org/10.21474/IJAR01/6306.

[8] O. Maxwell, B. A. Mayowa, I. U. Chinedu, A. E. Peace, Modelling count data: A Generalized Linear Model framework, American Journal of Mathematics and Statistics, **8**, no. 6, (2018), 179–183. http://dx.doi.org/10.5923/j.ajms.20180806.03.

[9] G. Bahk, H. Lee, Microbial-maximum likelihood estimation tool for microbial quantification in food from left-censored data using maximum likelihood estimation for microbial risk assessment, Frontiers in Microbiology, **12**, (2021). https://doi.org/10.3389/fmicb.2021.730733.

[10] J. Guinness, Gaussian process learning via Fisher Scoring of Vecchia's Approximation, Stat. Comput., **31**, no. 3, (2021), 1–9. https://doi.org/10.1007/s11222-021-09999-1.

[11] S. F. Mokhtar, Z. M. Yusof, H. Sapiri, Confidence intervals by Bootstrapping approach: a significance review, Malaysian Journal of Fundamental and Applied Sciences, **19**, (2023), 30–42. https://doi.org/10.11113/mjfas.v19n1.2660.