International Journal of Mathematics and
Computer Science, **19**(2024), no. 4, 1143–1145

Unsolvability of Two Diophantine Equations of the Form $p^a + (p-1)^b = c^2$

William S. Gayo Jr., Venus D. Siong

Don Mariano Marcos Memorial State University
La Union, Philippines

email: wgayo@dmmmsu.edu.ph, vsiong@dmmmsu.edu.ph

(Received March 9, 2024, Accepted April 24, 2024,
Published June 1, 2024)

Abstract

In this research study, we use elementary methods in number theory to show that the Diophantine equations $11^a + 10^b = c^2$ and $17^a + 16^b = c^2$ are unsolvable in non-negative integers.

1 Introduction

A Diophantine equation $f(x_1, x_2, \ldots, x_n) = 0$ is solvable if there exists ordered n-tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ that satisfies the given equation. These n-tuples are called its integer solutions. If no solution exists, the Diophantine equation is said to be unsolvable. Diophantine analysis seeks to answer whether a certain Diophantine equation is solvable or not.

Solvability of the equation of the form $p^a + (p-1)^b = c^2$ where p is a prime has been explored by few researchers as can be seen in [1] and [2] . In this study, we will show that the Diophantine equation $p^a + (p-1)^b = c^2$ is unsolvable in non-negative integers when $p = 11, 17$.

2 Preliminaries

The following theorem and lemmas are needed for the main result.

Key words and phrases: Diophantine equation, integer solutions, Mihailescu's Theorem, unsolvable.
AMS (MOS) Subject Classifications: 11D61.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

Theorem 2.1 (Mihailescu's Theorem). *[3] The quadruple* $(3, 2, 2, 3)$ *is the unique solution of the Diophantine equation* $x^a - y^b = 1$ *where* a, b, x *and* y *are integers with* $min\{a, b, x, y\} > 1$.

The following two lemmas are corollaries to Mihailescu's Theorem:

Lemma 2.2. *[4] The triple* $(3, 1, 2)$ *is the unique non-negative integer solution of the Diophantine equation* $p^a + 1 = c^2$ *where* p *is an odd prime.*

Lemma 2.3. *The triple* $(3, 3, 3)$ *is the unique non-negative integer solution of the Diophantine equation* $1 + (p - 1)^b = c^2$ *where* p *is a prime.*

Proof. If $b = 0$, then $c^2 = 2$ which has no integral solution. As a result, $b \geq 1$. It follows that $c^2 = 1 + (p - 1)^b \geq 1 + p - 1 = p > 1$. Thus $c > 1$. If $b = 1$, then $c^2 = p$ which is a contradiction. If $b > 1$, then by Mihailescu's Theorem, $p = 3, b = 3$ and $c = 3$. Thus, $(3, 3, 3)$ is the unique solution to $1 + (p - 1)^b = c^2$. □

The next two lemmas can be proven easily using modular techniques.

Lemma 2.4. *The square of an odd integer is congruent to* $1 \pmod 8$.

Lemma 2.5. *The square of an odd integer is congruent to* 1 *or* $3 \pmod 6$.

3 Main Results

Here, we discuss the main findings of our study.

Theorem 3.1. *The Diophantine equation* $11^a + 10^b = c^2$ *has no solution for non-negative integers.*

Proof. If $a = 0$ or $b = 0$, then we have the Diophantine equations $1 + 10^b = c^2$ and $11^a + 1 = c^2$ which have no non-negative solutions by Lemmas 2.3 and 2.2, respectively.

If $a, b > 0$, then c is odd. Now, note that $11^a \equiv 5 \pmod 6$ for odd integer a and $11^a \equiv 1 \pmod 6$ for even integer a. Also, $10^b \equiv 4 \pmod 6$ for any positive integer b. Since $c^2 \equiv 1, 3 \pmod 6$ by Lemma 2.5, a must be odd. Note also that $11^a \equiv 3 \pmod 8$ for odd integer b and $10^b \equiv 2 \pmod 8$ for $b = 1$, $10^b \equiv 4 \pmod 8$ for $b = 2$ and $10^b \equiv 0 \pmod 8$ for $b \geq 3$. Thus $11^a + 10^b \equiv 3, 5, 7 \pmod 8$. This is a contradiction because $c^2 \equiv 1 \pmod 8$. □

Theorem 3.2. *The Diophantine equation* $17^a + 16^b = c^2$ *has no solution for non-negative integers.*

Proof. To get a contradiction, suppose that there are non-negative integers a, b and c such that $17^a + 16^b = c^2$. If $a = 0$ or $b = 0$, then the Diophantine equations $1 + 16^b = c^2$ and $17^a + 1 = c^2$ have no non-negative solutions by Lemmas 2.3 and Lemma 2.2, respectively.

For $a, b > 0$, note that $c^2 - 16^b = (c - 4^b)(c + 4^b) = 17^a$. It follows that $c - 4^b = 17^\alpha$ and $c + 4^b = 17^{a-\alpha}$, where $a - \alpha > \alpha$. Subtracting the two equations gives $2 \cdot 4^b = 17^{a-\alpha} - 17^\alpha$ which can be expressed as $2^{2b+1} = 17^\alpha(17^{a-2\alpha} - 1)$. Then $\alpha = 0$ and $2^{2b+1} = 17^a - 1$. If $a = 1$, then $2^{2b+1} = 16$ which yields $b = 3/2$, a contradiction to b being an integer. If $a > 1$, then by Mihailescu's Theorem, it has no solution. ☐

4 Conclusion

Using modular arithmetic method, the factoring method, and Mihailescu's theorem we have shown that the Diophantine equations $11^a + 10^b = c^2$ and $17^a + 16^b = c^2$ have no non-integer solutions.

References

[1] W. S. Gayo Jr., J. B. Bacani, On the solutions Diophantine equation $M^x + (M-1)^y = z^2$, Italian Journal of Pure and Applied Mathematics, **14,** no. 2, (2022), 1113–1117.

[2] B. Sroysang, On the Diophantine equation $2^x + 3^y = z^2$, Int. J. Pure Appl. Math., **84,** no. 2, (2013), 133–137.

[3] P. Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, J. Reine Angew. Math., no. 27, (2004), 167–195.

[4] A. Suvarnamani, On the Diophantine equation $p^x + (p+1)^y = z^2$, Int. J. Pure Appl. Math., **95,** no. 4, (2014), 689–692.