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Weakly Pseudo Semi 2-Absorbing Submodule

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Abstract

Let R be a commutative ring with identity. In this paper, we introduce the notion of a weakly pseudo-semi-2-absorbing submodule as a generalization of a 2-absorbing submodule and a pseudo-2-absorbing submodule. Moreover, we give many basic properties, examples, and characterizations of these notions.

1 Introduction

In this paper, all rings are commutative with a nonzero identity and D is a unitary R-module. During the last 14 years, the notion of 2-absorbing submodules were investigated by Darani and Soheilinia [1] and that of semi-2-absorbing submodules by Naoum and Hasan [2]. In recent years, some generalizations have been made such as weakly semi-2-absorbing, pseudo-2-absorbing, weakly pseudo-2-absorbing, pseudo-semi-2-absorbing, weakly pseudo-2-absorbing, and pseudo-semi-2-absorbing submodules [3, 4, 5]. In section two of this paper, we study many properties of weakly pseudo-semi-2-absorbing submodules and explore the relations among these notions. In

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addition, we show that the intersection of two distinct weakly pseudo-semi-2-absorbing submodules is not necessarily a weakly pseudo-semi-2-absorbing submodule.

2 Main results

Definition 2.1. *A proper submodule S of an R–module D is said to be a weakly pseudo semi-2-absorbing (for short WPS-2AB) submodule of D if whenever $0 \neq r^2d \in S$, for $r \in R$, $d \in D$, we either have $rd \in S + Soc(D)$ or $r^2 \in [S + Soc(D) :_R D]$. An ideal Q of a ring R is said to be a WPS 2-AB ideal of R if Q is a weakly pseudo semi-2-absorbing submodule of an R-module R.*

Example 2.2. *In the $\mathbb{Z}$-module $\mathbb{Z}_{48}$, the submodules $(\overline{2}), (\overline{3}), (\overline{4}), (\overline{1}2)$ are WPS 2-AB submodules of $\mathbb{Z}_{48}$ and the submodules $(\overline{8}), (\overline{1}6), (\overline{2}4)$ of $\mathbb{Z}_{48}$ are not WPS 2-AB submodules.*

Proposition 2.3. *Let S be a submodule of an R-module D. Then*

1. *If S is weakly 2-absorbing, then S is WPS 2-AB. (The converse is not always true).*
2. *If S is weakly semi-2-absorbing, then S is WPS 2-AB. (The converse is not always true).*
3. *If S is weakly pseudo-2-absorbing, then S is WPS 2-AB. (The converse is not always true).*
4. *If S is pseudo-semi-2-absorbing, then S is a WPS 2-AB. (The converse is not always true).*

Proof.
(1) Assume that S is a 2-absorbing submodule of an R-module D and let $0 \neq r^2d \in S$, for $r \in R$, $d \in D$. Then $0 \neq rrd \in S$ and S is a 2-absorbing submodule. Therefore, either $rd \in S \subseteq S + Soc(D)$ or $r^2 \in [S :_R D]$; that is, $r^2D \subseteq S \subseteq S + Soc(D)$ and so S is a WPS 2-AB submodule. To show that the converse is not always true, in the $\mathbb{Z}$-module $\mathbb{Z}_{48}$, consider $(\overline{12})$ which is a WPS 2-AB submodule by example 2.2 but it is not 2-absorbing since $2.3.\overline{2} = \overline{12} \in (\overline{12})$, $2.\overline{2} = \overline{4} \notin (\overline{12})$, $3.\overline{2} = \overline{6} \notin (\overline{12})$, and $3.2 = 6 \notin [(\overline{12}) :_{\mathbb{Z}} \mathbb{Z}_{48}] = 12\mathbb{Z}$.
(2) Assume that S is a semi 2-absorbing submodule of an R-module D and let $0 \neq r^2d \in S$, for $r \in R$, $d \in D$. Then either $rd \in S \subseteq S + Soc(D)$

or $r^2 \in [S :_R D]$; that is, S is a WPS 2-AB submodule. To show that the converse is not always true, in the $\mathbb{Z}$-module $\mathbb{Z}_{48}$, consider $(\overline{12})$ which is a WPS 2-AB submodule by example 2.2 but it is not semi 2-absorbing since $2^2.\overline{3} = \overline{1}2 \in (\overline{12})$ $2.\overline{3} = \overline{6} \notin (\overline{12})$ and $2^2 = 4 \notin [(\overline{12}) :_{\mathbb{Z}} \mathbb{Z}_{48}] = 12\mathbb{Z}$.
(3) Assume that S is a pseudo 2-absorbing submodule of an R-module D and let $0 \neq r^2 d \in S$, for $r \in R$, $d \in D$. Then either $rd \in S + Soc(D)$ or $rr = r^2 \in [S + Soc(D) :_R D]$;that is, S is WPS 2-AB submodule. To show that the converse is not always true, in the $\mathbb{Z}$-module $\mathbb{Z}_{48}$, consider the submodule $(\overline{6})$ which is a WPS 2-AB submodule by example 2.2 but it is not a pseudo-2-absorbing since $2.3.\overline{1} = \overline{6} \in (\overline{6})$ for $2 \in \mathbb{Z}$, $\overline{1} \in \mathbb{Z}_{48}$. $2.\overline{1} = \overline{2} \notin (\overline{6}) + Soc(\mathbb{Z}_{48}) = (\overline{4})$, $3.\overline{1} = \overline{3} \notin (\overline{6}) + Soc(\mathbb{Z}_{48}) = (\overline{4})$. Also, $2.3 = 6 \notin [(\overline{6}) + Soc(\mathbb{Z}_{48}) :_{\mathbb{Z}} \mathbb{Z}_{48}] = [(\overline{4}) :_{\mathbb{Z}} \mathbb{Z}_{48}] = 4\mathbb{Z}$.
(4) This follows easily from the definition of WPS 2-AB. To show that the converse is not always true, consider $D = \mathbb{Z} \oplus \mathbb{Z}$ and the submodule $S = 4\mathbb{Z} \oplus (0)$. Clearly, S is WPS 2-AB but not pseudo semi 2-absorbing.

Lemma 2.4. *[6] $m\mathbb{Z}$ is a 2-absorbing submodule of the $\mathbb{Z}$–module $\mathbb{Z}$ if $m = 0$, $m = p_1$, $m = p_1p_2$, where p_1 and p_2 are prime numbers.*

Remark 2.5. *$m\mathbb{Z}$ is a WPS 2-AB submodule of the $\mathbb{Z}$ –module $\mathbb{Z}$ if $m = 0$, $m = p_1$, $m = p_1p_2$, where p_1 and p_2 are prime numbers.*

Proof.
By lemma 2.4, $m\mathbb{Z}$ is a 2-absorbing submodule. By proposition 2.3, every 2-absorbing submodule is a WPS 2-AB submodule. Consequently, $m\mathbb{Z}$ is a WPS 2-AB submodule.

Remark 2.6. *In general, the intersection of a pair of distinct WPS 2-AB submodules of D is not a WPS 2-AB submodule since $3\mathbb{Z}$ and $4\mathbb{Z}$ are WPS 2-AB submodules by remark 2.5 but $3\mathbb{Z} \cap 4\mathbb{Z} = 12\mathbb{Z}$ is not a WPS 2-AB submodule since $2^2 3 \in 12\mathbb{Z}$ but $2.3 \notin 12\mathbb{Z} + Soc(\mathbb{Z}) = 12\mathbb{Z} + (0) = 12\mathbb{Z}$ and $2^2 \notin [12\mathbb{Z} + Soc(\mathbb{Z}) :_{\mathbb{Z}} \mathbb{Z}] = [12\mathbb{Z} + (0) :_{\mathbb{Z}} \mathbb{Z}] = [12\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}] = 12\mathbb{Z}$.*

Remark 2.7. *Let S be a WPS 2-AB submodule of an R-Module D. $[S :_R D]$ is not necessarily a WPS 2-AB submodule because, for example, in the $\mathbb{Z}$-module $\mathbb{Z}_{32}$, $(\overline{8})$ is a WPS 2-AB submodule of $\mathbb{Z}_{32}$ but $[(\overline{8}) :_{\mathbb{Z}} \mathbb{Z}] = 8\mathbb{Z}$ is not a WPS 2-AB submodule since $0 \neq 2^2.2 = 8 \in 8\mathbb{Z}$ for $2 \in Z$, $2.2 = 4 \notin 8\mathbb{Z}$ and $2^2 \notin 8\mathbb{Z}$.*

Proposition 2.8. *Let S be a proper submodule of an R-module D. S is a WPS 2-AB submodule of an R-module D if and only if, for any $r \in R$ such that $r^2 \notin [S + Soc(D) :_R D]$, we have $[S :_D r^2] \subseteq [0 :_D r^2] \cup [S + Soc(D) :_D r]$.*

Proof.
Assume that S is a WPS 2-AB submodule and let $x \in [S :_D r^2]$. We prove that $x \in [0 :_D r^2] \cup [S + Soc(D) :_D r]$. If $0 \neq r^2x \in S$ and $r^2 \notin [S + Soc(D) :_R D]$, then $rx \in S + Soc(D)$ (since S is a WPS 2-AB submodule). Thus $x \in [S + Soc(D) :_D r]$. If $0 = r^2x \in S$, then $x \in [0 :_D r^2]$. Hence $x \in [0 :_D r^2] \cup [S + Soc(D) :_D r]$.
Conversely, assume that $0 \neq r^2d \in S$, for $r \in R$, $d \in D$ and let $[S :_D r^2] \subseteq [0 :_D r^2] \cup [S + Soc(D) :_D r]$ and $r^2 \notin [S + Soc(D) :_R D]$. Then, by our hypothesis, $d \notin [0 :_D r^2]$ but $d \in [S :_D r^2] \subseteq [0 :_D r^2] \cup [S + Soc(D) :_D r]$ since $d \notin [0 :_D r^2]$. Then $d \in [S + Soc(D) :_D r]$; that is, $rd \in S + Soc(D)$. Therefore, S is a WPS 2-AB submodule.

Proposition 2.9. *Let S be a proper submodule of an R-module D. Then S is a WPS 2-AB submodule of D if and only if $(0) \neq r^2T \subseteq S$ for $r \in R$ and T is submodule of D, it follows that either $rT \subseteq S + Soc(D)$ or $r^2 \in [S + Soc(D) :_R D]$.*

Proof.
Assume that S is a WPS 2-AB submodule of an R-module D, $(0) \neq r^2T \subseteq S$ for $r \in R$, and T is a submodule of D. Assume that $rT \nsubseteq S+Soc(D)$ and $r^2 \notin [S+Soc(D) :_R D]$. Then there exists $t \in T$ such that $rt \notin S+Soc(D)$. Now, since $0 \neq r^2t \in S$, $r^2 \notin [S+Soc(D) :_R D]$, and S is a WPS 2-AB submodule. Then, by proposition 2.8, $t \in [S :_D r^2] \subseteq [0 :_D r^2] \cup [S+Soc(D) :_D r]$ implies that $t \in [0 :_D r^2] \cup [S+Soc(D) :_D r]$. But $r^2t \neq 0$ and $rt \notin S+Soc(D)$; that is, $t \notin [0 :_D r^2]$ and $t \notin [S + Soc(D) :_D r]$ which is a contradiction. Hence $rT \subseteq S + Soc(D)$ or $r^2 \in [S + Soc(D) :_R D]$.
Conversely, assume that $0 \neq r^2d \in S$, for $r \in R$, $d \in D$. Then $(0) \neq r^2(d) \subseteq S$. Hence, by our hypothesis, $r^2(d) \subseteq S + Soc(D)$ or $r^2 \in [S + Soc(D) :_R D]$. Hence S is a WPS 2-AB submodule.

Proposition 2.10. *Let S be a proper submodule of an R-module D. Then S is a WPS 2-AB submodule of D if and only if $(0) \neq Q^2T \subseteq S$ for some ideal Q of R and submodule T of D which implies that either $QT \subseteq S + Soc(D)$ or $Q^2 \subseteq [S + Soc(D) :_R D]$.*

Proof.
Assume that S is a WPS 2-AB submodule of an R-module D and $(0) \neq Q^2T \subseteq S$, for some ideal Q of R and submodule T of D, and $Q^2 \nsubseteq [S + Soc(D) :_R D]$. To prove $QT \subseteq S + Soc(D)$, assume that $x \in QT$. This implies that $x = q_1t_1 + q_2t_2 + ... + q_it_i$ for $q_i \in Q$, $t_i \in T$ for all $i = 1, ..., n$. Thus ${q_i}^2t_i \in Q^2T \subseteq S$, for ${q_i}^2 \notin [S + Soc(D) :_R D]$, as S is a WPS 2-AB

submodule of D which implies that $q_i t_i \in S + Soc(D)$, for all $i = 1, ..., n$. Hence $x \in S + Soc(D)$. Consequently, $QT \subseteq S + Soc(D)$.
Conversely, assume that $0 \neq r^2 d \subseteq S$ for $r \in R$ and $d \in D$; that is, $0 \neq r^2 T \subseteq S$. Then, by hypothesis, either $(r)T \subseteq S + Soc(D)$ or $r^2 \subseteq [S + Soc(D) :_R D]$. Therefore, either $rT \subseteq S + Soc(D)$ or $r^2 \subseteq [S + Soc(D) :_R D]$. That is, S is a WPS 2-AB submodule of D.
The following corollary follows immediately from proposition 2.10.

Corollary 2.11. *Let S be a proper submodule of an R-module D, S is a WPS 2-AB submodule if and only if $(0) \neq Q^2 d \subseteq S$ for some ideal Q of R and $d \in D$, which implies that $Qd \subseteq S + Soc(D)$ or $Q^2 \subseteq [S + Soc(D) :_R D]$.*

Lemma 2.12. *[7] Let S, B and R be proper submodules of an R–module D with $B \subseteq R$. Then $(S + B) \cap R = (S \cap R) + B = (S \cap R) + (B \cap R)$.*

The following statement shows that, under certain circumstances, a WPS 2-AB submodule is the intersection of two WPS 2-AB submodules.

Proposition 2.13. *Let S and T be WPS 2-AB submodules of an R-module D with T not contained in S and either $Soc(D) \subseteq S$ or $Soc(D) \subseteq T$. Then $S \cap T$ is a WPS 2-AB submodule of D.*

$S \cap T$ is a proper submodule of T and T is a proper submodule of D. This implies that $S \cap T$ is a proper submodule of D. Assume that $Soc(D) \subseteq S$ But $Soc(D) \nsubseteq T$. Let $0 \neq r^2 Q \subseteq S \cap T$, for some $r \in R$ and Q is a submodule of T; that is, Q is a submodule of D. It follows that $0 \neq r^2 Q \subseteq S$ and $0 \neq r^2 Q \subseteq T$, But S and T are WPS 2-AB submodules. Then, by Proposition 2.10, either $rQ \subseteq S + Soc(D)$ or $r^2 D \subseteq S + Soc(D)$ and either $rQ \subseteq T + Soc(D)$ or $r^2 D \subseteq T + Soc(D)$. It follows that either $rQ \subseteq (S + Soc(D)) \cap T$ or $r^2 D \subseteq (S + Soc(D)) \cap T$ since $Soc(D) \subseteq T$. Then, by lemma 2.12, either $rQ \subseteq (S \cap T) + Soc(D)$ or $r^2 D \subseteq (S \cap T) + Soc(D)$. Hence $S \cap T$ is a WPS 2-AB submodule of D.

Lemma 2.14. *[7] Let M and N be R–modules and $f : M \to N$ be an R-epimorphism. Then $f(Soc(M)) \subseteq Soc(N)$.*

Proposition 2.15. *Let $f : M_1 \to M_2$ be an R-epimorphism and let Q be a WPS 2-AB submodule of M_1 with $ker(f) \subseteq Q$. Then $f(Q)$ is a WPS 2-AB submodule of M_2*

Proof.
Its clear that $f(Q)$ is submodule of M_2. Let $0 \neq r^2 m_2 \in f(Q)$, for some

$r \in R$ and $m_2 \in M_2$ and let $r^2 \notin [f(Q) + Soc(M_2) :_R M_2]$. We prove that $rm_2 \in f(Q) + Soc(M_2)$. Since f is onto, $0 \neq r^2 f(m_1) \in f(Q)$, for some $m_1 \in M_1$; that is, $f(r^2 m_1) = f(q)$ for some $q \in Q$. That is $r^2 m_1 - q \in ker(f) \subseteq Q$. This implies that $r^2 m_1 \in Q$ and so $m_1 \in [Q :_{M_1} r^2]$. By proposition 2.8, it follows that $m_1 \in [Q :_{M_1} r^2] \subseteq [0 : M_1 r^2] \cup [Q + Soc(M_1) :_{M_1} r]$. But Q is a WPS 2-AB submodule of M_1 so $r^2 m_1 \neq 0$; that is, $m_1 \notin [0 :_{M_1} r^2]$. Therefore, $m_1 \in [Q + Soc(M_1) :_{M_1} r]$; that is, $rm_1 \in Q + Soc(M_1)$ by lemma 2.14. So $rf(m_1) \in f(Q) + f(Soc(M_1) \subseteq f(Q) + Soc(M_2)$. Thus $rm_2 \in f(Q) + Soc(M_2)$. Consequently, $f(Q)$ is a WPS 2-AB submodule of M_2.

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