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Sum and Difference of Powers of Two Fibonacci Numbers

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Abstract

Let p be a prime number and let $x, k > 1$ be integers. We find all nonnegative integer solutions (n, m, x, p, k) to the Diophantine equations $F_n^x \pm F_m^x = p^k$ for $0 \leq m < n$, where F_n and F_m are the n-th and m-th Fibonacci numbers, respectively. For $m \neq 0$, the $\gcd(F_n, F_m) = 1$ and $F_n^x + F_m^x = p^k$, where x is not a power of 2.

1 Introduction

The Fibonacci sequence is a sequence of integers defined by $F_{n+2} = F_{n+1} + F_n$, where $F_0 = 0$ and $F_1 = 1$ for all integers $n \geq 0$. Many researchers have studied the Diophantine equation

$$F_n^x \pm F_m^x = y^a$$

which involves powers of Fibonacci numbers. Cohn [1] studied the case where $m = 0$ and $x = 1$ for the perfect square of Fibonacci numbers. This result was later extended by Bugeaud, Mignotte, and Siksek [2] for powers of Fibonacci numbers. Moreover, Bugeaud, Mignotte & Siksek [3], Kebli, Kihel, Larone,

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Luca [4], Bravo & Luca [5], Luca &Patel [6] and Ziegler [7] explored solutions for the said Diophantine equation when $m = 1$ or $m = 2$. Furthermore, Zhang and Togbe [8] have found results for the case when x is a prime number and $a \geq 2$. Patel and Chaves [9], Kohno & Luca[10], Luca & Oyono [11], Marques & Togbe [12] have investigated the case where y is a Fibonacci number. Finally, Taclay [13] has found nonzero integer solutions when $a = 2$. Building on the work of previous studies, in this paper we investigate the solutions (n, m, x, p, k) for the equation $F_n^x \pm F_m^x = p^k$.

2 Preliminaries

Before presenting the main results, we review some known theorems.

The following theorem is a result of Bugeaud et al. [2].

Theorem 2.1. *The only Fibonacci numbers F_n that are perfect powers are* 0, 1, 8, 144*; that is,* $n = 0, 1, 2, 6, 12$.

The next theorem is the well-known Zsigmondy's Theorem which can be found in [14].

Theorem 2.2. *If a, b and n are positive integers with $a > b$,* $\gcd(a, b) = 1$ *and $n \geq 2$, then $a^n - b^n$ has at least one prime factor that does not divide $a^k - b^k$ for all positive integers $k < n$, with the exceptions of*

- $2^6 - 1^6$ *and*
- $n = 2$ *and $a + b$ is a power of 2.*

Similarly, if a and b and n are positive integers with $a > b$ and $n \geq 2$, then $a^n + b^n$ has at least one prime factor that does not divide $a^k + b^k$ for all positive integers $k < n$, with the exception of $2^3 + 1^3$.

Finally, the following result can be found in [5].

Theorem 2.3. *The only solutions of the Diophantine equation $F_n + F_m = 2^a$ in positive integers n, m and a, with $1 \leq m < n$ are given by*

$$(n, m, a) \in \{(2, 1, 1), (4, 1, 2), (4, 2, 2), (5, 4, 3), (7, 4, 4)\}.$$

3 Main results

Theorem 3.1. *Let p be prime and $x, k > 1$. All the solutions of the Diophantine equation $F_n^x \pm F_m^x = p^k$ with $m = 0$, in nonnegative integers (n, m, x, p, k) are of the form $(\alpha, 0, k, F_\alpha, k)$ and $(6, 0, \frac{k}{3}, 2, k)$, where F_α is a Fibonacci prime, $k \in \mathbb{N}$ and $\frac{k}{3} \in \mathbb{N}$.*

Proof. Let p be prime, $x, k > 1$ and $m = 0$. We have $F_n^x = p^k$; that is, either $F_n = p$ or $F_n \neq p$.

Case 1. $F_n = p$. We can deduce that $x = k > 0$. Thus $(n, m, x, p, k) = (\alpha, 0, k, F_\alpha, k)$, where F_α is a Fibonacci prime and $k \in \mathbb{N}$.

Case 2. $F_n \neq p$. We have $F_n = p^\beta$, for some positive integer β. By Theorem (2.1), $F_n = 2^3$. Hence $(n, m, x, p, k) = (6, 0, \frac{k}{3}, 2, k)$, where $\frac{k}{3} \in \mathbb{N}$. □

Theorem 3.2. *Let p be prime and $x, k > 1$. For $F_n^x + F_m^x = p^k$, x is not a power of 2. All the solutions of the Diophantine equation $F_n^x \pm F_m^x = p^k$ with $\gcd(F_n, F_m) = 1$ and $1 \leq m < n$, for positive integers (n, m, x, p, k) are*

$$(4, 1, 2, 2, 3), (4, 2, 2, 2, 3), (5, 4, 2, 2, 4), (3, 1, 3, 3, 2), (3, 2, 3, 3, 2).$$

Proof. Let p be prime and $x, k > 1$. Consider the equation $F_n^x - F_m^x = p^k$. If $F_n^x - F_m^x$ has a prime factor that does not divide $F_n - F_m$, then, since $F_n^x - F_m^x$ is divisible by $F_n - F_m$, we can deduce that $F_n^x - F_m^x$ has at least two distinct prime factors. This contradicts Theorem (2.2), unless we encounter the exceptional cases where $x = 6$ and $F_n = 2$, $F_m = 1$, or where $x = 2$ and $F_n + F_m$ is a power of 2. However, the first case is not true. Therefore, $F_n + F_m = 2^\alpha$, for some positive integer α. By Theorem (2.3), $(n, m, x, p, k) = (4, 1, 2, 2, 3), (4, 2, 2, 2, 3), (5, 4, 2, 2, 4)$.

Similarly, for the equation $F_n^x + F_m^x = p^k$ and x is not a power of 2, we have the exceptional case $2^3 + 1^3$. Hence $(n, m, x, p, k) = (3, 1, 3, 3, 2), (3, 2, 3, 3, 2)$. □

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