Independence and Star Polynomials: Interrelations in Biclique Polynomials

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Abstract

In this paper, we establish the biclique polynomial of graphs resulting from the corona of two connected graphs in terms of the independence polynomials and star polynomials.

1 Introduction

The study of representing a graph in terms of a polynomial garnered interests recently since this representation captured applications in other fields of sciences such as Chemistry, Biology, and Physics [3]. Several discrete mathematicians established many results by considering specific subgraph structures of a given graph. Hoede and Li [4] investigated the independent

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set polynomial of graphs which counts the number of independent subsets of
the vertex-set of a graph of all possible cardinalities. Villarta et al. [7] es-
established the induced path polynomials in the join and the corona of graphs.
Moreover, Maldo and Artes [6] pioneered a work on geometric independence
polynomial of a graph and established a result using the Chuh-Shih-Chieh’s
Identity. The biclique polynomial was first introduced by Lumpayao et al.
[5]. This polynomial counts the number of bicliques in a graph of all orders.

2 Interrelations

It is interesting to note that stars are bicliques and the partite sets of bicliques
are independent sets. These facts give us ideas on the structure of graphs
resulting from the corona of two graphs and how to establish the biclique
polynomial in terms of star polynomial and independent polynomial of graphs
being considered in the operation.

A biclique in $G$ is a subset of $V(G)$ which induces a complete bipartite
graph in $G$. The biclique polynomial of a graph $G$, denoted by $\Gamma_b(G;x)$, is
given by $\Gamma_b(G;x) = \sum_{i=2}^{\beta(G)} b_i(G)x^i$, where $b_i(G)$ is the number of bicliques in $G$
of cardinality $i$ and $\beta(G)$ is the cardinality of the maximum biclique in $G$ [5].

A subset $S$ of $V(G)$ is an independent set in $G$ if the elements of $S$
are pairwise non-adjacent. The independence polynomial of $G$ is given by
$I(G;x) = \sum_{i=1}^{\alpha(G)} \alpha_i(G)x^i$, where $\alpha_i(G)$ is the number of independent subsets
of $V(G)$ of cardinality $i$ and $\alpha(G)$ is the independence number of $G$, the
maximum cardinality of an independent set in $G$ [4].

The star polynomial of a graph $G$ is defined as $\Gamma_s(G;x) = \sum_{i=1}^{s(G)} s_i(G)x^i$,
where $s_i(G)$ is the number of induced starts in $G$ of cardinality $i$ and $s(G)$
is the cardinality of the maximum induced star in $G$. Artes Jr. et al. estab-
lished results on special graphs and the corona of graphs ([1],[2]).

First, we characterize the bicliques in the corona of graphs. Given two
connected graphs $G$ and $H$, the corona $G \circ H$ of $G$ with $H$ has vertex-set
$V(G \circ H) = V(G) \cup \bigcup_{v \in V(G)} V(H_v)$, where $H_v$ is a copy of $H$ attached to
\[ v \in V(G), \text{ and edge-set} \]
\[ E(G \circ H) = E(G) \cup \bigcup_{v \in V(G)} [E(H_v) \cup \{uz : u \in V(G), z \in V(H_v)\}] \]

The adjacency in \( G \circ H \) carries the adjacency of \( G \) and the adjacency in \( H \) in every copy \( H_v \) of \( H \) and adding additional edges by joining each vertex \( v \in V(G) \) to every vertex of \( H_v \).

The bicliques in the corona of graphs are characterized in the following lemma.

**Lemma 2.1.** Suppose \( G \) and \( H \) are nontrivial connected graphs. A subset \( S \) of \( V(G \circ H) \) is a biclique in \( G \circ H \) if and only if it satisfies one of the following conditions:

(i) \( S \) is a biclique in \( G \)

(ii) \( S \) is a biclique in a copy of \( H \)

(iii) \( S = S_G \cup S_H \) where \( S_G \) induces a star in \( G \) and \( S_H \) is an independent set in \( H \).

**Proof.** Assume \( S \subseteq V(G \circ H) \) is a biclique in \( G \circ H \). Then \( S = S_1 \cup S_2 \), where \( S_1 \) and \( S_2 \) are the partite sets. We have the following cases:

**Case 1:** \( S \cap V(G) \neq \emptyset \).

**Subcase 1.1:** \( S \cap V(H_v) = \emptyset \) for every copy \( H_v \) of \( H \) attached to \( v \in V(G) \).

In this case, the set \( S \) is a biclique in \( G \) and \( (i) \) is satisfied.

**Subcase 1.2:** There exists a vertex in \( G \) satisfying \( S \cap V(H_v) \neq \emptyset \).

Then \( v \in S \). Note that if \( u \in V(G) \) and \( u \) is different from \( v \), then \( S \cap V(H_u) = \emptyset \) by biclique properties. Now, the only vertex in \( G \) adjacent to \( H_v \) is the vertex \( v \). This forces \( V(H_v) \cap S \) to be independent in \( H_v \) and \( S \cap V(G) \) must induce a star \( K_{1,|S \cap V(G)|-1} \) in \( G \) which is also a star in \( G \circ H \). Hence \( (iii) \) is satisfied.

**Case 2:** \( S \cap V(G) = \emptyset \).

Note that \( S \) can only intersect one copy of \( H \); say, \( H_v \), where \( v \in V(G) \). Thus \( S \) is a biclique in a copy of \( H \) and hence a biclique of \( H \). Thus \( (ii) \) is satisfied.

Conversely, any conditions in the lemma will imply that \( S \) is a biclique in \( G \circ H \). \( \square \)
Finally, we have the following theorem which establishes the biclique polynomial of the corona.

**Theorem 2.2.** Suppose that $G$ and $H$ are nontrivial connected graphs. Then

$$
\Gamma_b(G \circ H; x) = \Gamma_b(G; x) + |V(G)| \Gamma_b(H; x) + \Gamma_s(G; x) I(H; x).
$$

**Proof.** Let $S$ be a biclique in $G \circ H$. Then $S$ satisfies the conditions of Lemma 2.1. The first condition gives us the first term of the polynomial. The second condition gives us the second term of the polynomial. The last term of the polynomial follows from condition $(iii)$ of Lemma 2.1. \hfill \Box

The independence polynomial and star polynomial play significant roles in establishing our main result.

**References**


