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Bounds on Coefficients for a Subclass of Bi-Univalent Functions with Lucas-Balancing Polynomials and Ruscheweyh Derivative Operator

Abdulmtalb Hussen[1], **Moamar M. Alamari**[2]

[1]School of Engineering, Math., & Technology
Navajo Technical University
Crownpoint, NM 87313, USA

[2]Higher Institution of Engineering Technology
Zliten, Libya

email: ahussen@navajotech.edu

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Abstract

In this paper, we present a novel subclass of bi-univalent functions, which are connected with both the Ruscheweyh derivative operator and Lucas-Balancing polynomials. We establish bounds for the coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series for these functions, as well as the Fekete-Szegö inequality. Additionally, through parameter allocation in our primary discoveries, we unveil several fresh results.

1 Introduction

Let $\mathcal{A}$ represent the class of functions f of the form

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \tag{1.1}$$

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which are analytic in the open unit disk $\mathbb{U} = \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}$, and additionally satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$.

Given two functions f and g belonging to the class $\mathcal{A}$, we say that $f(\xi)$ is subordinate to $g(\xi)$ in the open unit disk $\mathbb{U}$, denoted as $f(\xi) \prec g(\xi)$, if there exists a Schwarz function $h(\xi)$, which is analytic in $\mathbb{U}$, satisfying the conditions $h(0) = 0$ and $|h(\xi)| < 1$ for all $\xi \in \mathbb{U}$, such that $f(\xi) = g(h(\xi))$ holds for all $\xi \in \mathbb{U}$. Moreover, if the function g is univalent in $\mathbb{U}$, then the following equivalence holds (referenced as [1]:

$$f(\xi) \prec g(\xi) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{S}$ be the set of all functions $f \in \mathcal{A}$ that are univalent within $\mathbb{U}$. According to the Koebe one-quarter theorem [2], for every function $f \in \mathcal{S}$, there exists an inverse function f^{-1} such that:

$$f^{-1}(f(\xi)) = \xi, \quad \xi \in \mathbb{U}$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f),\, r_0(f) \geq \frac{1}{4}$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and its inverse $g = f^{-1}$ are univalent in $\mathbb{U}$. Let σ denote the set of such bi-univalent functions in $\mathbb{U}$, as defined by equation (1.1). Recent studies have introduced various subclasses of σ, aiming to establish bounds for the first two coefficients, $|a_2|$ and $|a_3|$, in the Taylor-Maclaurin series expansion, as well as in the Fekete-Szegö inequality (see [3–13]).

The Hadamard product (or convolution) of $f(\xi)$ and $l(\xi)$, denoted as $f(\xi) * l(\xi)$, can be expressed mathematically as:

$$(f * l)(\xi) = \xi + \sum_{n=2}^{\infty} a_n b_n \xi^n = (l * f)(\xi) \quad (\xi \in \mathbb{U}),$$

where $l(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n$ is an analytic function in $\mathbb{U}$.

Definition 1.1. *[14] Let $f \in \mathcal{A}$ denote a function defined by equation* (1.1)*. The Ruscheweyh derivative operator $\mathcal{R}^\ell : \mathcal{A} \to \mathcal{A}$ is defined as follows:*

$$\mathcal{R}^\ell f(\xi) = \frac{\xi \left(\xi^{\ell-1} f(\xi)\right)^{(\ell)}}{\ell!} = \frac{\xi}{(1-\xi)^{\ell+1}} * f(\xi) = \xi + \sum_{n=2}^{\infty} \frac{\Gamma(\ell+n)}{\Gamma(n)\Gamma(\ell+1)} a_n \xi^n, \tag{1.3}$$

where $\ell \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}, \quad \xi \in \mathbb{U}$.

Behera and Panda [15] recently introduced a novel integer sequence known as Balancing numbers. These numbers are generated by the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$, with initial values $B_0 = 0$ and $B_1 = 1$. This introduction has sparked significant interest among researchers, leading to the exploration of various generalizations. For comprehensive insights into Lucas-Balancing numbers and their extensions, refer to the works cited in [16–24]. Among these extensions, one notable example is the Lucas Balancing polynomial, which is recursively defined as follows:

Definition 1.2. *[25] For every complex numbe t and integer $n \geq 2$, Lucas-Balancing polynomials are recursively defined as such:*

$$C_n(t) = 6tC_{n-1}(t) - C_{n-2}(t), \tag{1.4}$$

where the initial conditions are given by:

$$C_0(t) = 1, \quad C_1(t) = 3t. \tag{1.5}$$

By employing the recurrence relation (1.4), we can derive the subsequent expressions:

$$C_2(t) = 18t^2 - 1 \quad C_3(t) = 108t^3 - 9t. \tag{1.6}$$

Lucas-Balancing polynomials, similar to other number polynomials, can be obtained using specific generating functions. An example of such a generating function is represented as follows:

Lemma 1.3. *[25] The generating function for Balancing polynomials can be represented as*

$$\mathcal{B}(t, \xi) = \sum_{n=0}^{\infty} C_n(t)\xi^n = \frac{1 - 3t\xi}{1 - 6t\xi + \xi^2}, \tag{1.7}$$

where t is within the range $[-1, 1]$, and ξ is in the open unit disk $\mathbb{U}$.

2 Coefficient Bounds of the Class $\mathcal{H}_\sigma(\alpha, \mu, \mathcal{B}(t, \xi))$

Definition 2.1. *Let $f \in \sigma$ be given by (1.1), with $\alpha, \mu \in [0, 1]$ and $t \in \left(\frac{1}{2}, 1\right]$. We say f is in the class $\mathcal{H}_\sigma(\alpha, \mu, \mathcal{B}(t, \xi))$ if the following subordinations are satisfied:*

$$(1-\alpha+2\mu)\frac{\mathcal{R}^\ell f(\xi)}{\xi} + (\alpha-2\mu)\left(\mathcal{R}^\ell f(\xi)\right)' + \mu\xi\left(\mathcal{R}^\ell f(\xi)\right)'' \prec \mathcal{B}(t, \xi) \quad (2.8)$$

and

$$(1-\alpha+2\mu)\frac{\mathcal{R}^\ell g(w)}{w} + (\alpha-2\mu)\left(\mathcal{R}^\ell g(w)\right)' + \mu w\left(\mathcal{R}^\ell g(w)\right)'' \prec \mathcal{B}(t, w), \quad (2.9)$$

where the function $g(w) = f^{-1}(w)$ is defined by (1.2) and $\mathcal{B}(t, \xi)$ represents the generating function of the Lucas-Balancing polynomials as given by equation (1.7).

Example 2.1. *Let $f \in \sigma$ be a bi-univalent function. It is said to belong to the class $\mathcal{H}_\sigma(\alpha, 0, \mathcal{B}(t, \xi))$ if the following subordination conditions hold:*

$$(1-\alpha)\frac{\mathcal{R}^\ell f(\xi)}{\xi} + \alpha\left(\mathcal{R}^\ell f(\xi)\right)' \prec \mathcal{B}(t, \xi) \quad (2.10)$$

and

$$(1-\alpha)\frac{\mathcal{R}^\ell g(w)}{w} + \alpha\left(\mathcal{R}^\ell g(w)\right)' \prec \mathcal{B}(t, w), \quad (2.11)$$

where the function $g = f^{-1}$ is defined by (1.2).

Example 2.2. *Let $f \in \sigma$ be a bi-univalent function. It is said to belong to the class $\mathcal{H}_\sigma\big(1, 0, \mathcal{B}(t, \xi)\big)$, if the following subordination conditions hold:*

$$\left(\mathcal{R}^\ell f(\xi)\right)' \prec \mathcal{B}(t, \xi) \quad (2.12)$$

and

$$\left(\mathcal{R}^\ell g(w)\right)' \prec \mathcal{B}(t, w), \quad (2.13)$$

where the function $g = f^{-1}$ is defined by (1.2).

Lemma 2.2. *[2] Let Ω be the class of all analytic functions, and let $\omega \in \Omega$ with $\omega(\xi) = \sum_{n=1}^{\infty} \omega_n \xi^n$, $\xi \in \mathbb{D}$. Then,*

$$|\omega_1| \leq 1, \quad |\omega_n| \leq 1 - |\omega_1|^2 \quad for \quad n \in \mathbb{N}\backslash\{1\}.$$

Theorem 2.3. *Let $f \in \sigma$ of the form (1.1) be in the class $\mathcal{H}_\sigma(\alpha, \mu, \mathcal{B}(t, \xi))$. Then*

$$|a_2| \leq \frac{3t\Gamma(2)\Gamma(\ell+1)\sqrt{3t\Gamma(3)}}{\sqrt{\left|9t^2\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu) - (18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2\right|}}, \tag{2.14}$$

and

$$|a_3| \leq \frac{27t^3\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2}{\left|9t^2\Gamma(\ell+1)\Gamma(\ell+3)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu) - (18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2\right|} + \frac{3t\Gamma(3)\left(\Gamma(\ell+1)\right)^2}{\Gamma(\ell+3)(1+2\alpha+2\mu)}. \tag{2.15}$$

Proof. Let $f \in \mathcal{H}_\sigma(\alpha, \mu, \mathcal{B}(t, \xi))$ for some $0 \leq \alpha, \mu \leq 1$, and from (2.8) and (2.9) we have

$$(1-\alpha+2\mu)\frac{\mathcal{R}^\ell f(\xi)}{\xi} + (\alpha - 2\mu)\left(\mathcal{R}^\ell f(\xi)\right)' + \mu\xi\left(\mathcal{R}^\ell f(\xi)\right)'' = \mathcal{B}(t, \xi) \tag{2.16}$$

and

$$(1-\alpha+2\mu)\frac{\mathcal{R}^\ell g(w)}{w} + (\alpha - 2\mu)\left(\mathcal{R}^\ell g(w)\right)' + \mu w\left(\mathcal{R}^\ell g(w)\right)'' = \mathcal{B}(t, w), \tag{2.17}$$

where $g(w) = f^{-1}(w)$ and $u, v \in \Omega$ are given to be of the form

$$u(\xi) = \sum_{n=1}^{\infty} c_n \xi^n \qquad and \qquad v(w) = \sum_{n=1}^{\infty} d_n w^n.$$

From Lemma 2.2, we have

$$|c_n| \leq 1 \ \text{ and } \ |d_n| \leq 1, \ n \in \mathbb{N}. \tag{2.18}$$

Upon substituting the definition of $\mathcal{B}(t,\xi)$ from (1.7) into the right-hand sides of equations (2.16) and (2.17), we obtain

$$\begin{aligned}\mathcal{B}(t,u(\xi)) = 1 &+ C_1(t)c_1\xi + \left[C_1(t)c_2 + C_2(t)c_1^2\right]\xi^2 \\ &+ \left[C_1(t)c_3 + 2C_2(t)c_1c_2 + C_3(t)c_1^3\right]\xi^3 + \cdots,\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\mathcal{B}(t,v(w)) = 1 &+ C_1(t)d_1w + \left[C_1(t)d_2 + C_2(t)d_1^2\right]w^2 \\ &+ \left[C_1(t)d_3 + 2C_2(t)d_1d_2 + C_3(t)d_1^3\right]w^3 + \cdots.\end{aligned} \tag{2.20}$$

Therefore, equations (2.16) and (2.17) become

$$\begin{aligned}&1 + \frac{\Gamma(\ell+2)}{\Gamma(2)\Gamma(\ell+1)}(1+\alpha)a_2\xi + \frac{\Gamma(\ell+3)}{\Gamma(3)\Gamma(\ell+1)}(1+2\alpha+2\mu)a_3\xi^2 \\ &+ \frac{\Gamma(\ell+4)}{\Gamma(4)\Gamma(\ell+1)}(1+3\alpha+6\mu)a_4\xi^3 + \cdots \\ &= 1 + C_1(t)c_1\xi + \left[C_1(t)c_2 + C_2(t)c_1^2\right]\xi^2 + \left[C_1(t)c_3 + 2C_2(t)c_1c_2 + C_3(t)c_1^3\right]\xi^3 + \cdots,\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&1 - \frac{\Gamma(\ell+2)}{\Gamma(2)\Gamma(\ell+1)}(1+\alpha)a_2w + \frac{\Gamma(\ell+3)}{\Gamma(3)\Gamma(\ell+1)}\Big[2(1+2\alpha+2\mu)a_2^2 - (1+2\alpha+2\mu)a_3\Big]w^2 \\ &+ \frac{\Gamma(\ell+4)}{\Gamma(4)\Gamma(\ell+1)}\Big[\left(20\left(2\mu-\alpha\right) - 5\left(2\mu-\alpha+1\right) - 60\mu\right)a_2^3 \\ &+ \left(-20\left(2\mu-\alpha\right) + 5\left(2\mu-\alpha+1\right) + 60\mu\right)a_2a_3 + \left(-6\mu-3\alpha-1\right)a_4\Big]w^3 + \cdots \\ &= 1 + C_1(t)d_1w + \left[C_1(t)d_2 + C_2(t)d_1^2\right]w^2 + \left[C_1(t)d_3 + 2C_2(t)d_1d_2 + C_3(t)d_1^3\right]w^3 + \cdots.\end{aligned} \tag{2.22}$$

Upon equating the coefficients in equations (2.21) and (2.22), we obtain:

$$\frac{\Gamma(\ell+2)}{\Gamma(2)\Gamma(\ell+1)}(1+\alpha)a_2 = C_1(t)c_1, \tag{2.23}$$

$$\frac{\Gamma(\ell+3)}{\Gamma(3)\Gamma(\ell+1)}(1+2\alpha+2\mu)a_3 = C_1(t)c_2 + C_2(t)c_1^2, \tag{2.24}$$

$$-\frac{\Gamma(\ell+2)}{\Gamma(2)\Gamma(\ell+1)}(1+\alpha)a_2 = C_1(t)d_1, \tag{2.25}$$

and

$$\frac{\Gamma(\ell+3)}{\Gamma(3)\Gamma(\ell+1)}\Big[2(1+2\alpha+2\mu)a_2^2-(1+2\alpha+2\mu)a_3\Big] = C_1(t)d_2+C_2(t)d_1^2. \tag{2.26}$$

By employing equations (2.23) and (2.25), we deduce the subsequent expressions:

$$c_1 = -d_1 \tag{2.27}$$

and

$$c_1^2 + d_1^2 = \frac{2\big(\Gamma(\ell+2)\big)^2(1+\alpha)^2a_2^2}{\big(\Gamma(2)\Gamma(\ell+1)\big)^2\big(C_1(t)\big)^2}. \tag{2.28}$$

Moreover, employing equations (2.24), (2.26), and (2.28) yields:

$$a_2^2 = \frac{\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2\big(C_1(t)\big)^3(c_2+d_2)}{2\left[\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu)(C_1(t))^2 - \Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2C_2(t)\right]}. \tag{2.29}$$

By employing Lemma 2.2 and analyzing equations (2.23) and (2.27), we have

$$|a_2|^2 \leq \frac{\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2|C_1(t)|^3}{\left|\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu)(C_1(t))^2 - \Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2C_2(t)\right|}, \tag{2.30}$$

therefore

$$|a_2| \leq \frac{\Gamma(2)\Gamma(\ell+1)\,|C_1(t)|\sqrt{\Gamma(3)\,|C_1(t)|}}{\sqrt{\left|\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu)(C_1(t))^2 - \Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2C_2(t)\right|}}. \tag{2.31}$$

Substituting $C_1(t)$ and $C_2(t)$, as given in equations (1.5) and (1.6) respectively, into equation (2.31) yields the subsequent expression,

$$|a_2| \leq \frac{3t\Gamma(2)\Gamma(\ell+1)\sqrt{3t\Gamma(3)}}{\sqrt{\left|9t^2\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu)-(18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2\right|}}.$$

Subtracting equation (2.26) from equation (2.24) yields:

$$a_3 = a_2^2 + \frac{\Gamma(3)\Gamma(\ell+1)C_1(t)(c_2-d_2)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}. \tag{2.32}$$

As a result, this leads to the subsequent inequality:

$$|a_3| \leq |a_2|^2 + \frac{\Gamma(3)\Gamma(\ell+1)\,|C_1(t)|\,|c_2-d_2|}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}. \tag{2.33}$$

By applying Lemma 2.2 and employing equations (1.5) and (1.6), we derive:

$$|a_3| \leq \frac{27t^3\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2}{\left|9t^2\Gamma(\ell+1)\Gamma(\ell+3)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu)-(18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2\right|} + \frac{3t\Gamma(3)\left(\Gamma(\ell+1)\right)^2}{\Gamma(\ell+3)(1+2\alpha+2\mu)}. \tag{2.34}$$

The proof of Theorem 2.3 is now complete. □

3 Fekete–Szegö Functional Estimations of the Class $\mathcal{H}_\sigma\big(\alpha,\mu,\mathcal{B}(t,\xi)\big)$

In this section, utilizing the values of a_2^2 and a_3 aids in deriving the Fekete–Szegö inequality applicable to functions within the domain of $\mathcal{H}_\sigma\big(\alpha,\mu,\mathcal{B}(t,\xi)\big)$.

Theorem 3.1. *Let $f \in \sigma$ given by the form (1.1) be in the class $\mathcal{H}_\Sigma(\alpha,\mu,\mathcal{B}(t,\xi))$. Then*

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{3t\Gamma(3)\Gamma(\ell+1)}{\Gamma(\ell+3)(1+2\alpha+2\mu)} & \text{if} \quad 0 \leq |h(\eta)| \leq \frac{\Gamma(3)\Gamma(\ell+1)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)} \\ 6t\,|h(\eta)| & \text{if} \quad |h(\eta)| \geq \frac{\Gamma(3)\Gamma(\ell+1)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}, \end{cases}$$

where

$$h(\eta)=\frac{9t^2\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2(1-\eta)}{2\left[9t^2\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu)-(18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2\right]}.$$

Proof. Equations (2.29) and (2.32) yield

$$\begin{aligned}
a_3-\eta a_2^2 &= a_2^2+\frac{\Gamma(3)\Gamma(\ell+1)C_1(t)(c_2-d_2)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}-\eta a_2^2\\
&=(1-\eta)a_2^2+\frac{\Gamma(3)\Gamma(\ell+1)C_1(t)(c_2-d_2)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}\\
&=(1-\eta)\frac{\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2\left(C_1(t)\right)^3(c_2+d_2)}{2\left[\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu)(C_1(t))^2-\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2C_2(t)\right]}\\
&+\frac{\Gamma(3)\Gamma(\ell+1)C_1(t)(c_2-d_2)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}\\
&=\left(C_1(t)\right)\left(\left[h(\eta)+\frac{\Gamma(3)\Gamma(\ell+1)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}\right]c_2+\left[h(\eta)-\frac{\Gamma(3)\Gamma(\ell+1)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}\right]d_2\right),
\end{aligned}$$

where

$$h(\eta)=\frac{\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2\left(C_1(t)\right)^2(1-\eta)}{2\left[\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha+2\mu)(C_1(t))^2-\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2C_2(t)\right]}.$$

By considering equations (1.5) and (1.6), and applying equation (2.18), we can conclude that

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\frac{3t\Gamma(3)\Gamma(\ell+1)}{\Gamma(\ell+3)(1+2\alpha+2\mu)} & \text{if}\quad 0\le|h(\eta)|\le\frac{\Gamma(3)\Gamma(\ell+1)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}\\ 6t\,|h(\eta)| & \text{if}\quad |h(\eta)|\ge\frac{\Gamma(3)\Gamma(\ell+1)}{2\Gamma(\ell+3)(1+2\alpha+2\mu)}.\end{cases}$$

The proof of Theorem 3.1 is now complete. □

Corollary 3.2. *Let $f\in\sigma$ given by the form (1.1) be in the class $\mathcal{H}_\sigma\big(\alpha,0,\mathcal{B}(t,\xi)\big)$. Then*

$$|a_2| \leq \frac{3t\Gamma(2)\Gamma(\ell+1)\sqrt{3t\Gamma(3)}}{\sqrt{\left|9t^2\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha)-(18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2\right|}},$$

$$|a_3| \leq \frac{27t^3\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2}{\left|9t^2\Gamma(\ell+1)\Gamma(\ell+3)\left(\Gamma(2)\right)^2(1+2\alpha)-(18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2\right|} + \frac{3t\Gamma(3)\left(\Gamma(\ell+1)\right)^2}{\Gamma(\ell+3)(1+2\alpha)},$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{3t\Gamma(3)\Gamma(\ell+1)}{\Gamma(\ell+3)(1+2\alpha)} & if \quad 0 \leq |h_1(\eta)| \leq \frac{\Gamma(3)\Gamma(\ell+1)}{2\Gamma(\ell+3)(1+2\alpha)} \\ 6t\,|h_1(\eta)| & if \quad |h_1(\eta)| \geq \frac{\Gamma(3)\Gamma(\ell+1)}{2\Gamma(\ell+3)(1+2\alpha)}, \end{cases}$$

where

$$h_1(\eta) = \frac{9t^2\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2(1-\eta)}{2\left[9t^2\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2(1+2\alpha)-(18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2(1+\alpha)^2\right]}.$$

Corollary 3.3. *Let $f \in \sigma$ given by the form (1.1) be in the class ,$\mathcal{H}_\sigma\big(1,0,\mathcal{B}(t,\xi)\big)$. Then*

$$|a_2| \leq \frac{3t\Gamma(2)\Gamma(\ell+1)\sqrt{3t\Gamma(3)}}{\sqrt{\left|27t^2\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2-4(18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2\right|}},$$

$$|a_3| \leq \frac{27t^3\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2}{\left|27t^2\Gamma(\ell+1)\Gamma(\ell+3)\left(\Gamma(2)\right)^2-4(18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2\right|} + \frac{t\Gamma(3)\left(\Gamma(\ell+1)\right)^2}{\Gamma(\ell+3)},$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{t\Gamma(3)\Gamma(\ell+1)}{\Gamma(\ell+3)} & if \quad 0 \leq |h_2(\eta)| \leq \frac{\Gamma(3)\Gamma(\ell+1)}{6\Gamma(\ell+3)} \\ 6t\,|h_2(\eta)| & if \quad |h_2(\eta)| \geq \frac{\Gamma(3)\Gamma(\ell+1)}{6\Gamma(\ell+3)}, \end{cases}$$

where

$$h_2(\eta) = \frac{9t^2\Gamma(3)\left(\Gamma(2)\Gamma(\ell+1)\right)^2(1-\eta)}{2\left[27t^2\Gamma(\ell+3)\Gamma(\ell+1)\left(\Gamma(2)\right)^2 - 4(18t^2-1)\Gamma(3)\left(\Gamma(\ell+2)\right)^2\right]}.$$

4 Conclusions

In this paper, we have introduced and explored a new subclass of analytic bi-univalent functions denoted as $\mathcal{H}_\sigma(\alpha, \mu, \mathcal{B}(t, \xi))$, which are linked to Lucas-Balancing Polynomials and the Ruscheweyh derivative operator. Our investigation focuses on initial estimates of Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Moreover, using a_2^2 and a_3, we establish Fekete-Szegö inequalities for functions in this subclass. Moreover, By specializing parameters, we established connections between subclass, Lucas-Balancing Polynomials and Ruscheweyh derivative operator, deriving estimates for Taylor-Maclaurin coefficients and exploring Fekete-Szegö inequalities.

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