International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 731–738

# The Partial Algebra of Terms with a Fixed Number of Variables under a Generalized Superposition

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(Received December 3, 2023, Accepted January 17, 2024,
Published February 12, 2024 )

**Abstract**

In this paper, we focus on terms with fixed variables count, terms under which the total numbers of occurrences of variables in each position are equal. Moreover, we determine conditions for which the set of terms with fixed variables count is closed under the generalized superposition. Furthermore, we form the partial algebras of such terms satisfying certain axioms as weak identities.

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**Key words and phrases:** Term, partial operation, weak identity.
**AMS (MOS) Subject Classifications**: 08A05, 03D05, 03B15.
The corresponding author is Thawhat Changphas.
**ISSN** 1814-0432, 2024, http://ijmcs.future-in-tech.net

# 1 Introduction and preliminaries

Following [3], we recall that a term of type $\tau$ is an expression arising from variables from an infinite set of alphabet $X = \{x_1, x_2, \ldots\}$ or a finite set $X_n = \{x_1, x_2, \ldots, x_n\}$ for a positive integer $n$ and a family $\{f_i \mid i \in I\}$ of operation symbols for the arity $n_i$ for each $i$ in an index set $I$. By the definition, an $n$-ary term of type $\tau$ is inductively defined by the following steps: each variable $x_i$ in $X_n$ is a term of type $\tau$ and $f_i(t_1, \ldots, t_{n_i})$ is also a term of type $\tau$ if $t_1, \ldots, t_{n_i}$ are already known. The set $W_\tau(X_n)$ is the smallest set consisting of $n$-ary terms of type $\tau$ which is closed under finite application of each operation symbol $f_i$. Moreover, the symbol $W_\tau(X)$ denotes the set of all terms of type $\tau$ which means that $W_\tau(X) =: \cup_{n\geq 1} W_\tau(X_n)$. In particular, we say that terms in $W_\tau(X)$ are constructed from $X$. For more backgrounds of terms, see [2, 5, 7, 9].

In [8], the generalized superposition operation $S_g^n$ was applied to the set $W_\tau(X)$ of all terms of type $\tau$. By the definition, it is an operation of type $(n+1)$ inductively defined on $W_\tau(X)$ by the following steps: for $t, t_1, \ldots, t_n \in W_\tau(X)$

(1) If $t = x_i$; $1 \leq i \leq n$, then $S_g^n(x_i, t_1, \ldots, t_n) := t_i$.

(2) If $t = x_i$; $i > n$, then $S_g^n(x_i, t_1, \ldots, t_n) := x_i$.

(3) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S_g^n(t, t_1, \ldots, t_n) := f_i(S_g^n(s_1, t_1, \ldots, t_n), \ldots, S_g^n(s_{n_i}, t_1, \ldots, t_n))$.

As a consequence, the algebra $(W_\tau(X), S_g^n)$ of type $(n+1)$ was formed. Puninagool and Leeratanavalee [8] proved that this algebra belongs to the class of superassociative algebras or Menger algebras, see [4], because $S_g^n$ satisfies the superassociative law:

(C$_1$) $S_g^n(S_g^n(t, t_1, \ldots, t_n), s_1, \ldots, s_n)$
$= S_g^n(t, S_g^n(t_1, s_1, \ldots, s_n), \ldots, S_g^n(t_n, s_1, \ldots, s_n))$.

Moreover, adding a family $(x_i)_{i\geq 1}$, the superassociative algebra of terms with infinitely many nullary operations $(W_\tau(X), S_g^n, (x_j)_{j\geq 1})$ of type $(n+1, 0, 0, 0, \ldots)$ was obtained. This algebra satisfies (C1) and the following three equations:

(C2) $S_g^n(x_j, s_1, \ldots, s_n) = s_j$ if $1 \leq j \leq n$,

(C3) $S_g^n(x_j, s_1, \ldots, s_n) = x_j$ if $j > n$, (C4) $S^n_g(t, x_1, \ldots, x_n) = t$.

Following [1], recall that an equation $s \approx t$ is said to be a *weak identity* in an algebra $\mathcal{A}$ if one side is defined, then another side is also defined and both sides are equal. Moreover, by a partial superassociative algebra of rank $n$, we mean a pair of a nonempty set $M$ and a partial operation $\bar{o} : M^{n+1} \multimap\!\!\rightarrow M$ defined on $M$ such that an arbitrary condition $\bar{o}(a, b_1, \ldots, b_n)$ for elements $a, b_1, \ldots, b_n$ in $M$ is defined and $\bar{o}$ satisfies the superassociative law as a weak identity. It is clear that any partial semigroup is a partial superassociative algebra of rank 1. The partial algebraic structures of terms have been widely studied in this decade. For example, the partial Menger algebra of linear terms which are terms under which each variable that appears in a term occurs only once was introduced by Denecke [3]. Recently, Kumduang and Leeratanavalee [6] presented the partial algebra of terms with fixed variables count. To study such terms, one needs the formula for counting the complextity of terms by considering the number of variables that appear in a term.

In this paper, we apply the generalized superposition to the set of terms fixing the number of variables and demonstrate the process of computation. Conditions for which the set of such terms is closed under $S^n_g$ for a positive integer $n$ are determined. Based on the formula $\mathrm{vb}(S^n_g(s, t_1, \ldots, t_n)) = \sum_{j=1}^{n} \mathrm{vb}_j(s)\mathrm{vb}(t_j) + \sum_{j>n} \mathrm{vb}_j(s)$ given in [8], we prove that our structure satisfies some important axioms.

# 2 Results

We begin by recalling the measurement of the complexity of terms. The *variables count of a term* $t$ is the total number of occurring variables in $t$, and is denoted by $\mathrm{vb}(t)$. If $t$ is a variable, then $\mathrm{vb}(t) = 1$ and if $t = f_i(t_1, \ldots, t_{n_i})$, then $\mathrm{vb}(t) = \sum_{j=1}^{n_i} \mathrm{vb}(t_j)$.

**Definition 2.1.** *A term with fixed variables count of type $\tau$ is inductively defined by:*

*(1) Every variable $x_i$ in $X$ is a term with fixed variables count of type $\tau$.*

*(2) If $t_1, \ldots, t_{n_i}$ are terms with fixed variables count of type $\tau$, and if $\mathrm{vb}(t_k) = \mathrm{vb}(t_l)$ for all $1 \leq k < l \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is a term with fixed variables count of type $\tau$.*

(3) *The set* $W_\tau^{fvc}(X)$ *is the smallest set closed under finite application of* (2).

**Example 2.2.** *We consider the type* $\tau = (3,2)$ *with one ternary operation symbol and one binary operation symbol, say* $f$ *and* $g$ *respectively. Then*

$$x_1, x_4, x_{11}, f(x_7, x_2, x_4), f(g(x_2, x_2), g(x_4, x_2), g(x_5, x_9)) \in W_{(3,2)}^{fvc}(X),$$

*but*

$$f(g(x_2, x_8), x_4, x_1), g(x_1, f(x_1, x_2, x_4)) \notin W_{(3,2)}^{fvc}(X).$$

We observe that the set $W_\tau^{fvc}(X)$ of terms with fixed variables count of type $\tau$ is not closed under the generalized superposition in general. Consider the following example. On the set $W_{(3,2)}^{fvc}(X)$ of terms with fixed variables count corresponding to a ternary operation symbol $f$ and a binary operation symbol $g$, let $t = g(x_7, x_2), s_1 = x_3, s_2 = f(x_1, x_1, x_4)$. Then $S_g^2(t, s_1, s_2) = g(x_7, f(x_1, x_1, x_4))$ is not a term with fixed variables count although $t, s_1$ and $s_2$ are terms with fixed variables count. For this reason, it is possible to determine the conditions for which the set $W_\tau^{fvc}(X)$ is closed under $S_g^n$.

We give the following statements.

($A1$) $t$ is a variable from $X$ and $s_1, \ldots, s_n$ are terms with fixed variable of type $\tau$.

($A2$) $s_1, \ldots, s_n$ are elements in $W_\tau^{fvc}(X)$ and $t$ in $W_\tau^{fvc}(X) \setminus X$ such that $\text{var}(t) \subseteq X_n$ and $\text{vb}(s_l) = \text{vb}(s_m)$ for $1 \leq l < m \leq n$.

($A3$) $s_1, \ldots, s_n$ are elements in $W_\tau^{fvc}(X)$ and $t$ in $W_\tau^{fvc}(X) \setminus X$ such that $\text{var}(t) \cap X_n = \emptyset$.

($A4$) $s_1, \ldots, s_n$ are elements in $W_\tau^{fvc}(X)$ and $t$ in $W_\tau^{fvc}(X) \setminus X$ for which there exists $p > n$ such that $x_p \in \text{var}(t)$ and $\text{vb}(s_j) = 1$ for all $1 \leq j \leq n$.

We have:

**Lemma 2.3.** *If the condition* ($A1$) *holds, then* $S_g^n(t, s_1, \ldots, s_n) \in W_\tau^{fvc}(X)$.

*Proof.* The proof follows from the definition of $S_g^n$. □

**Lemma 2.4.** *Let* $s_1, \ldots, s_n$ *be elements in* $W_\tau^{fvc}(X)$ *and* $t$ *in* $W_\tau^{fvc}(X) \setminus X$. *Then the following statements are true:*

1. *If the condition* ($A1$) *holds, then* $S_g^n(t, s_1, \ldots, s_n) \in W_\tau^{fvc}(X)$.

2. *If the condition* $(A2)$ *holds, then* $S_g^n(t, s_1, \dots, s_n) \in W_\tau^{fvc}(X)$.

3. *If the condition* $(A3)$ *holds, then* $S_g^n(t, s_1, \dots, s_n) \in W_\tau^{fvc}(X)$.

*Proof.* Let $t = f_i(t_1, \dots, t_{n_i}), s_1, \dots, s_n$ be terms with fixed variables count of type $\tau$. Assume first that all variables of $t$ come from $X_n$ and $\mathrm{vb}(s_1) = \dots = \mathrm{vb}(s_n)$. By definition of $S_g^n$, we replace terms from $\{s_1, \dots, s_n\}$ for the variables occurring in each $t$. Thus $S_g^n(t, s_1, \dots, s_n)$ is also a term with fixed variables count of the same type. Suppose that $\mathrm{var}(t) \cap X_n = \emptyset$, which means that all variables in $t$ are in $X \setminus X_n$. It follows from the definition of $S_g^n$ that $S_g^n(t, s_1, \dots, s_n)$ equals $t$. Finally, assume that there exists $p > n$ such that $x_p \in \mathrm{var}(t)$ and $\mathrm{vb}(s_1) = \dots = \mathrm{vb}(s_n) = 1$. From the definition of the generalized superposition $S_g^n$, it is not difficult to see that the variable $x_p$ in $t$ cannot be replaced by any other terms from $\{s_1, \dots, s_n\}$. As a result, the condition $\mathrm{vb}(s_1) = \dots = \mathrm{vb}(s_n) = 1$ implies that $S_g^n(t, s_1, \dots, s_n)$ is again a term with fixed variables count. $\square$

Consequently, Lemmas 2.3 and 2.4 allow us to define the partial operation on $W_\tau^{fvc}(X)$. In fact, for a positive integer $n$ and $t, s_1, \dots, s_n \in W_\tau^{fvc}(X)$, the partial mapping

$$\overline{S}_g^n : (W_\tau^{fvc}(X))^{n+1} \multimap\!\!\rightarrow W_\tau^{fvc}(X)$$

is inductively defined by:

$$\overline{S}_g^n(t, s_1, \dots, s_n) := \begin{cases} S_g^n(t, s_1, \dots, s_n) & \text{, if } (A_1) \text{ or } (A_2) \text{ or } (A_3) \text{ or } (A_4) \text{ holds,} \\ \text{not defined} & \text{, otherwise.} \end{cases}$$

Therefore, the partial algebra $(W_\tau^{fvc}(X), \overline{S}_g^n)$ is formed.

We now show that the partial operation $\overline{S}_g^n$ defined for this partial algebra is superassociative as a weak identity.

**Theorem 2.5.** $(W_\tau^{fvc}(X), \overline{S}_g^n)$ *is a partial superassociative algebra.*

*Proof.* To show that the partial operation $\overline{S}_g^n$ satisfies $\overline{S}_g^n(\overline{S}_g^n(t, s_1, \dots, s_n), u_1 \dots, u_n) = \overline{S}_g^n(t, \overline{S}_g^n(s_1, u_1, \dots, u_n), \dots, \overline{S}_g^n(s_n, u_1, \dots, u_n))$ as a weak identity, for every $j = 1, \dots, n$, we let $t, s_j, u_j$ be elements in $W_\tau^{fvc}(X)$. Assume first that $\overline{S}_g^n(\overline{S}_g^n(t, s_1, \dots, s_n), u_1 \dots, u_n)$ is defined. Thus, we have the following cases:

**Case 1**: $t \in X$ and $S_g^n(t, s_1, \dots, s_n) \in X$.

**Case 2**: $t \in W_\tau^{fvc}(X) \setminus X, \mathrm{var}(t) \subseteq X_n$ and $\mathrm{vb}(s_l) = \mathrm{vb}(s_m)$ for $1 \leq l < m \leq n$ and $\mathrm{var}(S_g^n(t, s_1, \ldots, s_n)) \subseteq X_n$ and $\mathrm{vb}(u_l) = \mathrm{vb}(u_m)$ for $1 \leq l < m \leq n$.
**Case 3**: $t \in W_\tau^{fvc}(X) \setminus X, \mathrm{var}(t) \cap X_n = \emptyset$ and $\mathrm{var}(S_g^n(t, s_1, \ldots, s_n)) \cap X_n = \emptyset$.
**Case 4**: $t \in W_\tau^{fvc}(X) \setminus X$, there exists $p > n$ such that $x_p \in \mathrm{var}(t)$ and $\mathrm{vb}(s_j) = 1$ for all $1 \leq j \leq n$ and there exists $q > n$ such that $x_q \in \mathrm{var}(S_g^n(t, s_1, \ldots, s_n))$ and $\mathrm{vb}(u_j) = 1$ for all $1 \leq j \leq n$.

In the first case, $\overline{S}_g^n(\overline{S}_g^n(t, s_1, \ldots, s_n), u_1 \ldots, u_n)$ equals $S_g^n(s_i, u_1 \ldots, u_n)$ if $t = x_i \in X_n$ and equals $S_g^n(t, u_1 \ldots, u_n)$ and so $t$ if $t = x_i \in X \setminus X_n$. For $t = x_i \in X_n$, on the right-hand side, by the definition of $S_g^n$, we have $S_g^n(s_i, u_1 \ldots, u_n)$. Thus, in this case, our aim is obtained. If $t = x_i \in X \setminus X_n$, on the right-hand side, we get $t$. As a result, our goal is achieved.

For the second case, we have that $\overline{S}_g^n(\overline{S}_g^n(t, s_1, \ldots, s_n), u_1 \ldots, u_n)$ is equal to $S_g^n(S_g^n(t, s_1, \ldots, s_n), u_1 \ldots, u_n)$. Moreover, for each $1 \leq j \leq n$, $\overline{S}_g^n(s_j, u_1, \ldots, u_n)$ is also defined and equals $S_g^n(s_1, u_1, \ldots, u_n)$. Obviously, $S_g^n(s_1, u_1, \ldots, u_n) \in W_\tau^{fvc}(X)$. To show that $\mathrm{vb}(S_g^n(s_l, u_1, \ldots, u_n)) = \mathrm{vb}(S_g^n(s_m, u_1, \ldots, u_n))$ for $1 \leq l < m \leq n$, we see that $\mathrm{vb}(s_l) = \mathrm{vb}(s_m)$ for $1 \leq l < m \leq n$ and thus which means $\sum_{i=1}^n \mathrm{vb}_i(s_l) = \sum_{i=1}^n \mathrm{vb}_i(s_m)$. Because $\mathrm{vb}(u_l) = \mathrm{vb}(u_m)$ for $1 \leq l < m \leq n$, we obtain $\sum_{i=1}^n \mathrm{vb}_i(s_l)\mathrm{vb}(u_i) + \sum_{j>n} \mathrm{vb}_j(s_l) = \sum_{i=1}^n \mathrm{vb}_i(s_m)\mathrm{vb}(u_i) + \sum_{j>n} \mathrm{vb}_j(s_m)$. As a result, $\mathrm{vb}(S_g^n(s_l, u_1, \ldots, u_n)) = \mathrm{vb}(S_g^n(s_m, u_1, \ldots, u_n))$. Therefore, $\overline{S}_g^n(t, \overline{S}_g^n(s_1, u_1, \ldots, u_n), \ldots, \overline{S}_g^n(s_n, u_1, \ldots, u_n))$ is defined and is equal to $S_g^n(t, S_g^n(s_1, u_1, \ldots, u_n), \ldots, S_g^n(s_n, u_1, \ldots, u_n))$. Due to the superassociativity of $S_g^n$ given in [8], so our aim is obtained.

We now consider Case (3). In this case, $\overline{S}_g^n(\overline{S}_g^n(t, s_1, \ldots, s_n), u_1 \ldots, u_n)$ equals $t$. On the other hand, $\overline{S}_g^n(t, \overline{S}_g^n(s_1, u_1, \ldots, u_n), \ldots, \overline{S}_g^n(s_n, u_1, \ldots, u_n))$ is also defined and equals $t$. The proof of this case is complete.

The proof of Case (4) can be directly obtained from the definition of $S_g^n$ for $n \geq 1$. □

Considering a family $(x_i)_{i \geq 1}$ of variables which acts as a family of nullary operations, we have the following theorem:

**Theorem 2.6.** *$(W_\tau^{fvc}(X), \overline{S}_g^n, (x_i)_{i \geq 1})$ is a partial generalized superassociative algebra.*

*Proof.* It is enough to prove that the partial operation $\overline{S}_g^n$ defined on $W_\tau^{fvc}(X)$ satisfies the following three equations as weak identities: (1) $\overline{S}_g^n(x_j, s_1, \ldots, s_n) = s_j$ if $1 \leq j \leq n$, (2) $\overline{S}_g^n(x_j, s_1, \ldots, s_n) = x_j$ if $j > n$ and (3) $\overline{S}_g^n(t, x_1, \ldots, x_n) = t$. To prove (1), we let $s_1, \ldots, s_n$ be terms with fixed variables count of type $\tau$. For each $j = 1, \ldots, n$, assume that $\overline{S}_g^n(x_j, s_1, \ldots, s_n)$ is defined.

Then $\mathrm{var}(x_j) \subseteq X_n$ and $\mathrm{vb}(s_l) = \mathrm{vb}(s_m)$ for $1 \le l < m \le n$. Thus $\overline{S}^n_g(x_j, s_1, \ldots, s_n)$ equals $S^n_g(x_j, s_1, \ldots, s_n)$, which means $s_j$. If $j > n$ and $\overline{S}^n_g(x_j, s_1, \ldots, s_n)$ is defined, then we get $S^n_g(x_j, s_1, \ldots, s_n) = x_j$. Finally, suppose that for a term with fixed variables count $t$, $\overline{S}^n_g(t, x_1, \ldots, x_n)$ is defined. From this, by the conditions $(A1)$, $(A2)$, $(A3)$, and $(A4)$, we inductively divide our consideration into two parts. If $t = x_j$ for $1 \le j \le n$, then by $(A1)$, we have that $\overline{S}^n_g(x_j, x_1, \ldots, x_n)$ is equal to $S^n_g(x_j, x_1, \ldots, x_n)$ and thus $x_j$. If $t$ is a variable from $X \setminus X_n$, then by $(A2)$, our goal is obtained. If $t = f_i(t_1, \ldots, t_{n_i})$ and assume that each $t_k$ already known for $k = 1, \ldots, n_i$, then $\overline{S}^n_g(f_i(t_1, \ldots, t_{n_i}), x_1, \ldots, x_n)$ equals $S^n_g(f_i(t_1, \ldots, t_{n_i}), x_1, \ldots, x_n)$. By our inductive step, we also conclude that $S^n_g(f_i(t_1, \ldots, t_{n_i}), x_1, \ldots, x_n)$ and $f_i(t_1, \ldots, t_{n_i})$ are equal. $\square$

**Example 2.7.** *Consider the type $\tau = (2)$ with a binary operation symbol $f$ and a subset*

$$D = \{x_1, f(x_2, x_4), f(x_3, x_1), f(x_1, x_1)\}$$

*of $W^{fvc}_{(2)}(X)$ with respect to the partial operation $\overline{S}^1_g$ of an arity 2 defined by the following table:*

| $\overline{S}^1_g$ | $x_1$ | $f(x_2, x_4)$ | $f(x_3, x_1)$ | $f(x_1, x_1)$ |
|---|---|---|---|---|
| $x_1$ | $x_1$ | $f(x_2, x_4)$ | $f(x_3, x_1)$ | $f(x_1, x_1)$ |
| $f(x_2, x_4)$ | $f(x_2, x_4)$ | $f(x_2, x_4)$ | $f(x_2, x_4)$ | $f(x_2, x_4)$ |
| $f(x_3, x_1)$ | $f(x_3, x_1)$ | not defined | not defined | not defined |
| $f(x_1, x_1)$ | $f(x_1, x_1)$ | $f(f(x_2, x_4), f(x_2, x_4))$ | $f(f(x_3, x_1), f(x_3, x_1))$ | $f(f(x_1, x_1), f(x_1, x_1))$ |

It is not hard to see that the partial binary operation $\overline{S}^1_g$ defined on $D$ is weak associative. We give a demonstration as follows:
We consider terms $x_1, f(x_2, x_4)$ and $f(x_1, x_1)$ in $D$. To show that an equation

$$\overline{S}^1_g(\overline{S}^1_g(f(x_1, x_1), x_1), f(x_2, x_4)) = \overline{S}^1_g(f(x_1, x_1), \overline{S}^1_g(x_1, f(x_2, x_4)))$$

is a weak identity, suppose first that the left-hand side is defined. Hence, we have that $\overline{S}^1_g(\overline{S}^1_g(f(x_1, x_1), x_1), f(x_2, x_4))$ is equal to $S^1_g(f(x_1, x_1), f(x_2, x_4))$ and thus $f(f(x_2, x_4), f(x_2, x_4))$. As a result, the right-hand side is also defined and equals to $S^1_g(f(x_1, x_1), f(x_2, x_4))$.

**Acknowledgment.** This research was supported by the Department of Mathematics, Faculty of Science, Khon Kaen University, Fiscal Year 2022.

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