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Characterizations of fuzzy Bd-ideals in Bd-algebras

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Abstract

In 2022, Bantaojai et al. [3] introduced an algebra structure called Bd-algebras. In this paper, we define a new notion called fuzzy Bd-ideals of Bd-algebras and study some of its basic properties. Moreover, we characterize fuzzy Bd-ideals by the different types of their level subsets in Bd-algebras.

1 Introduction

The notion of BCK-algebras was introduced by Iséki and Tanaka [5]. Later, the notion of BCI-algebras was presented by Iséki [4], who showed that the

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class of BCI-algebras is an extension of BCK-algebras. In 1999, Neggers and Kim [7] presented d-algebras as an additional algebraic structure that generalizes the BCK-algebras. They also examined the relationships between d-algebras and BCK-algebras. Subsequently, they presented the class of abstract algebras, known as B-algebras, [8]. In 2022, some requirements from B-algebras and d-algebras were combined by Bantaojai et al. [3] to develop a new algebra, known as Bd-algebra, which provides some characteristics of Bd-ideals and Bd-subalgebras of Bd-algebras.

The notion of fuzzy sets was first defined by Zadeh [10] as a mapping from a nonempty set X to the unit interval $[0,1]$. Then, Jun et al. [6] considered the concepts of fuzzy B-subalgebras and fuzzy normals in B-algebras and characterized fuzzy B-subalgebras in B-algebras. Moreover, Baghini and Saeid [2] introduced the concept of (α, β)-fuzzy B-algebras, as a generalization fuzzy B-subalgebras in B-algebras. Afterwards, fuzzy subalgebras and fuzzy d-ideals in d-algebras were introduced by Akram and Dar [1], who additionally delved into some of their properties. In 2023, the notions of left and right fuzzy derivations of d-ideals of d-algebras were introduced by Oli and Tefera [9] who discussed many characterizations of the left and right fuzzy derivations of d-ideals of d-algebras.

The objective of this research is to apply the notion of fuzzy sets to study the algebraic structure of the Bd-algebras. We introduce the notion of fuzzy Bd-ideals in Bd-algebras and investigate the relationships between fuzzy Bd-ideals and Bd-ideals using their level subsets in Bd-algebras.

2 Preliminaries

A mapping $\mu : X \to [0,1]$ from a nonempty set X to a unit interval is known as a *fuzzy set* [10]. Let μ and λ be any two fuzzy sets of a nonempty set X. We use the following notation:

(i) $\mu^c(x) = 1 - \mu(x)$, for all $x \in X$;

(ii) $(\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\}$, for all $x \in X$;

(iii) $(\mu \cup \lambda)(x) = \max\{\mu(x), \lambda(x)\}$, for all $x \in X$.

The *characteristic function* χ_A of a subset A of a nonempty set X is a fuzzy set of X defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

for each $x \in X$.

Let μ be a fuzzy set of a nonempty set X. Then, for every $t \in [0, 1]$, we have (i) the set $U(\mu, t) = \{x \in X \mid \mu(x) \geq t\}$ is called an *upper t-level subset of μ*; (ii) the set $U^+(\mu, t) = \{x \in X \mid \mu(x) > t\}$ is called an *upper t-strong level subset of μ*; (iii) the set $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ is called a *lower t-level subset of μ*; (iv) the set $L^-(\mu, t) = \{x \in X \mid \mu(x) < t\}$ is called a *lower t-strong level subset of μ*.

Definition 2.1. *[3] An algebra $(X, *, 0)$ of type $(2, 0)$ is called a Bd-algebra if it satisfies the following axioms, for every $x, y \in X$: (i) $x * 0 = x$; (ii) if $x * y = 0$ and $y * x = 0$, then $x = y$.*

Throughout this paper, unless specified otherwise, we use the symbol X instead of a Bd-algebra $(X, *, 0)$.

Definition 2.2. *[3] A nonempty subset I of a Bd-algebra X is said to be a Bd-ideal of X if it satisfies the following conditions: (i) $0 \in I$; (ii) if for any $x, y \in X$, $x * y \in I$ and $y \in I$, then $x \in I$; (iii) $x * y \in I$, for all $x \in I$ and $y \in X$.*

3 Fuzzy Bd-ideals of Bd-algebras

In this section, we introduce the notion of fuzzy Bd-ideals of Bd-algebras and characterize fuzzy Bd-ideals of Bd-algebras based on some properties of their level subsets.

Definition 3.1. *Let X be a Bd-algebra. A fuzzy set μ of X is called a fuzzy Bd-ideal of X if, for every $x, y \in X$, it satisfies the following inequalities*

(i) $\mu(0) \geq \mu(x)$;

(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$;

(iii) $\mu(x * y) \geq \mu(x)$.

Example 3.2. *Let $X = \{0, a, b, c\}$. Define the binary operation $*$ on X as follows*

$*$	0	a	b	c
0	0	a	a	a
a	a	a	a	a
b	b	b	b	b
c	c	a	a	a

Then $(X, *, 0)$ is a Bd-algebra [3]. Now, we define the fuzzy set μ of X by $\mu(a) \leq \mu(c) \leq \mu(b) \leq \mu(0)$. It turns out that μ is a fuzzy Bd-ideal of X.

Proposition 3.3. *If μ and λ are fuzzy Bd-ideals of a Bd-algebra X, then $\mu \cap \lambda$ is also a fuzzy Bd-ideal of X.*

Proof. Assume that μ and λ are fuzzy Bd-ideals of a Bd-algebra X. For any $a \in X$, we have $(\mu \cap \lambda)(0) = \min\{\mu(0), \lambda(0)\} \geq \min\{\mu(a), \lambda(a)\} = (\mu \cap \lambda)(a)$. Let $x, y \in X$. Then, we have $(\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\} \geq \min\{\min\{\mu(x * y), \mu(y)\}, \min\{\lambda(x * y), \lambda(y)\}\} = \min\{\min\{\mu(x * y), \lambda(x * y)\}, \min\{\mu(y), \lambda(y)\}\} = \min\{(\mu \cap \lambda)(x * y), (\mu \cap \lambda)(y)\}$. In addition, $(\mu \cap \lambda)(x * y) \min\{\mu(x * y), \lambda(x * y)\} \geq \min\{\mu(x), \lambda(x)\} = (\mu \cap \lambda)(x)$. This shows that $\mu \cap \lambda$ is a fuzzy Bd-ideal of X. ☐

Example 3.4. *Let $X = \{0, a, b, c\}$ be a set with the binary operation $*$ on X defined by the following table*

$*$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	b	b	b	b
c	c	c	a	a

It is simple to verify that $(X, *, 0)$ is a Bd-algebra. Next, the fuzzy sets μ and λ of X are defined by

$$\mu(0) = 0.7, \mu(a) = 0.5, \mu(b) = 0.3, \mu(c) = 0.3,$$
$$\lambda(0) = 0.8, \lambda(a) = 0.4, \lambda(b) = 0.6, \lambda(c) = 0.4.$$

By routine calculations, we have μ and λ are fuzzy Bd-ideals of X. However, $\mu \cup \lambda$ is not a fuzzy Bd-ideal of X because $(\mu \cup \lambda)(c) = 0.4 < 0.5 = \min\{(\mu \cup \lambda)(c * b), (\mu \cup \lambda)(b)\}$.

From Example 3.4, we conclude that the union of fuzzy Bd-ideals of a Bd-algebra X may not be a fuzzy Bd-ideal of X.
Next, the proof of the following lemma is straightforward.

Lemma 3.5. *Let A be a nonempty subset of a Bd-algebra X. Then, A contains the element 0 of X if and only if χ_A satisfies $\chi_A(0) \geq \chi_A(x)$, for all $x \in X$.*

Theorem 3.6. *Let X be a Bd-algebra and let A be a nonempty subset of X. Then A is a Bd-ideal of X if and only if χ_A is a fuzzy Bd-ideal of X.*

Proof. Assume that A is a Bd-ideal of X. Since $0 \in A$, by Lemma 3.5 we have $\chi_A(0) \geq \chi_A(x)$, for all $x \in X$. Let $x, y \in X$. If $x \in A$, then $\chi_A(x) = 1 \geq \min\{\chi_A(x * y), \chi_A(y)\}$. If $x \notin A$, then $x * y \notin A$ or $y \notin A$. This implies that $\chi_A(x) = 0 = \min\{\chi_A(x * y), \chi_A(y)\}$. Next, suppose that $\chi_A(a * b) < \chi_A(a)$, for some $a, b \in X$. Then, $\chi_A(a * b) = 0$ and $\chi_A(a) = 1$; that is, $a * b \notin A$ and $a \in A$. Since A is a Bd-ideal of X, we have $a * b \in A$. This is a contradiction. So $\chi_A(x * y) \geq \chi_A(x)$, for all $x, y \in X$. Therefore, χ_A is a fuzzy Bd-ideal of X.

Conversely, assume that χ_A is a fuzzy Bd-ideal of X. By Lemma 3.5, $0 \in A$. Let $x, y \in X$ be such that $x * y \in A$ and $y \in A$. Then $\chi_A(x * y) = 1$ and $\chi_A(y) = 1$. It turns out that $\chi_A(x) \geq \min\{\chi_A(x * y), \chi_A(y)\} = 1$, and so $\chi_A(x) = 1$. That is, $x \in A$. Now, let $a \in A$ and $b \in X$. Thus, $\chi_A(a * b) \geq \chi_A(a) = 1$. It follows that $\chi_A(a * b) = 1$. Hence, $a * b \in A$. Consequently, A is a Bd-ideal of X. □

Theorem 3.7. *Let μ be a fuzzy set of a Bd-algebra X. Then μ is a fuzzy Bd-ideal of X if and only if, for every $t \in [0, 1]$, $U(\mu, t) \neq \emptyset$ is a Bd-ideal of X.*

Proof. Assume that μ is a fuzzy Bd-ideal of X. Let $t \in [0, 1]$ be such that $U(\mu, t) \neq \emptyset$. For any $a \in U(\mu, t)$, we have $\mu(0) \geq \mu(a) \geq t$. So $0 \in U(\mu, t)$. Next, let $x, y \in X$ be such that $x * y \in U(\mu, t)$ and $y \in U(\mu, t)$. Thus, $\mu(x * y) \geq t$ and $\mu(y) \geq t$. By assumption, we have $\mu(x) \geq \min\{\mu(x*y), \mu(y)\} \geq t$. This implies that $x \in U(\mu, t)$. Now, let $a \in U(\mu, t)$ and $b \in X$. It follows that $\mu(a * b) \geq \mu(a) \geq t$; that is, $a * b \in U(\mu, t)$. This shows that $U(\mu, t)$ is a Bd-ideal of X.

Conversely, assume that, for every $t \in [0, 1]$, $U(\mu, t) \neq \emptyset$ is a Bd-ideal of X. Let $x, y \in X$. We take $\mu(x) = t_1$, for some $t_1 \in [0, 1]$. Then, $x \in U(\mu, t_1)$. It follows that $U(\mu, t_1) \neq \emptyset$. We obtain $U(\mu, t_1)$ is a Bd-ideal of X. Thus $0 \in U(\mu, t_1)$. So, $\mu(0) \geq t_1 = \mu(x)$. Letting $t_2 = \min\{\mu(x * y), \mu(y)\}$, for some $t_2 \in [0, 1]$, $\mu(x * y) \geq t_2$ and $\mu(y) \geq t_2$. We have $x * y, y \in U(\mu, t_2)$. By assumption, we get $x \in U(\mu, t_2)$. Then $\mu(x) \geq t_2 = \min\{\mu(x * y), \mu(y)\}$. Next, choose $\mu(x) = t_3$, for some $t_3 \in [0, 1]$. Thus, $x \in U(\mu, t_3)$. Therefore, $\mu(x * y) \geq t_3 = \mu(x)$. Consequently, μ is a fuzzy Bd-ideal of X. □

Theorem 3.8. *Let μ be a fuzzy set of a Bd-algebra X. Then, μ is a fuzzy Bd-ideal of X if and only if, for every $t \in [0, 1]$, $U^+(\mu, t) \neq \emptyset$ is a Bd-ideal of X.*

Proof. Assume that μ is a fuzzy Bd-ideal of X. Let $t \in [0, 1]$ be such that $U^+(\mu, t) \neq \emptyset$ and let $w \in U^+(\mu, t)$. Then, $\mu(w) > t$. So, $\mu(0) \geq \mu(w) > t$.

Thus, $0 \in U^+(\mu, t)$. Next, let $x, y \in X$ be such that $x * y \in U^+(\mu, t)$ and $y \in U^+(\mu, t)$. Then, $\mu(x * y) > t$ and $\mu(y) > t$. By the hypothesis, we have $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} > t$ which implies that $x \in U^+(\mu, t)$. Now, let $a \in U^+(\mu, t)$ and $b \in X$. We obtain $\mu(a * b) \geq \mu(a) > t$. This means that $a * b \in U^+(\mu, t)$. Hence, $U^+(\mu, t)$ is a Bd-ideal of X.

Conversely, for every $t \in [0, 1]$, $U^+(\mu, t) \neq \emptyset$ is a Bd-ideal of X. Let $x, y \in X$. Suppose that $\mu(0) < \mu(x)$. Then, $x \in U^+(\mu, \mu(0))$. By assumption, we have $U^+(\mu, \mu(0))$ is a Bd-ideal of X. Thus, $0 \in U^+(\mu, \mu(0))$. So, $\mu(0) > \mu(0)$, which is a contradiction. Therefore, $\mu(0) \geq \mu(x)$. Now, if $\mu(x) < \min\{\mu(x * y), \mu(y)\}$, then $\mu(x * y) > \mu(x)$ and $\mu(y) > \mu(x)$. It turns out that $x * y, y \in U^+(\mu, \mu(x))$. So, $U^+(\mu, \mu(x))$ is a Bd-ideal of X. Also, $x \in U^+(\mu, \mu(x))$. It follows that $\mu(x) > \mu(x)$, which is a contradiction. Hence, $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Similarly, suppose that $\mu(x * y) < \mu(x)$. Thus, $x \in U^+(\mu, \mu(x * y))$. By the given assumption, we get $x * y \in U^+(\mu, \mu(x * y))$. Then, $\mu(x * y) > \mu(x * y)$. This is a contradiction. Hence, $\mu(x * y) \geq \mu(x)$. Therefore, μ is a fuzzy Bd-ideal of X. □

Theorem 3.9. *Let μ be a fuzzy set of a Bd-algebra X. Then, μ^c is a fuzzy Bd-ideal of X if and only if, for every $t \in [0, 1]$, $L(\mu, t) \neq \emptyset$ is a Bd-ideal of X.*

Proof. Assume that μ^c is a fuzzy Bd-ideal of X. Let $t \in [0, 1]$ be such that $L(\mu, t) \neq \emptyset$ and let $w \in L(\mu, t)$. Then, $\mu(w) \leq t$. Since $\mu^c(0) \geq \mu^c(w)$, we have $1 - \mu(0) \geq 1 - \mu(w)$. Also, $\mu(0) \leq \mu(w) \leq t$. Thus, $0 \in L(\mu, t)$. Next, let $x * y \in L(\mu, t)$ and $y \in L(\mu, t)$. Then, $\mu(x * y) \leq t$ and $\mu(y) \leq t$. By the hypothesis, we have $\mu^c(x) \geq \min\{\mu^c(x * y), \mu^c(y)\}$, which implies that $1 - \mu(x) \geq \min\{1 - \mu(x * y), 1 - \mu(y)\} = 1 - \max\{\mu(x * y), \mu(y)\}$. We see that $\mu(x) \leq \max\{\mu(x * y), \mu(y)\} \leq t$. So $x \in L(\mu, t)$. Finally, let $a \in L(\mu, t)$ and $b \in X$. It follows that $\mu^c(a * b) \geq \mu^c(a)$. As a result, $1 - \mu(a * b) \geq 1 - \mu(a)$. This shows that $\mu(a * b) \leq \mu(a) \leq t$. Hence, $a * b \in L(\mu, t)$. Therefore, $L(\mu, t)$ is a Bd-ideal of X.

Conversely, assume that, for every $t \in [0, 1]$, $L(\mu, t) \neq \emptyset$ is a Bd-ideal of X. Let $x, y \in X$. We take $\mu(x) = t_1$, for some $t_1 \in [0, 1]$. Then, $x \in L(\mu, t_1)$. So, $L(\mu, t_1)$ is a Bd-ideal of X. It follows that $0 \in L(\mu, t_1)$; that is, $\mu(0) \leq t_1 = \mu(x)$. Thus, $\mu^c(0) = 1 - \mu(0) \geq 1 - \mu(x) = \mu^c(x)$. Next, we choose $t_2 = \max\{\mu(x * y), \mu(y)\}$, for some $t_2 \in [0, 1]$. It follows that $\mu(x * y) \leq t_2$ and $\mu(y) \leq t_2$ which implies that $x * y \in L(\mu, t_2)$ and $y \in L(\mu, t_2)$. By the given assumption, we have $x \in L(\mu, t_2)$; that is, $\mu(x) \leq t_2 = \max\{\mu(x * y), \mu(y)\}$. We get $\mu^c(x) = 1 - \mu(x) \geq 1 - \max\{\mu(x * y), \mu(y)\} = \min\{1 - \mu(x * y), 1 - \mu(y)\} = \min\{\mu^c(x * y), \mu^c(y)\}$. Now, let $\mu(x) = t_3$, for some $t_3 \in [0,1]$. Then $x \in L(\mu, t_3)$. By the given hypothesis, we have $L(\mu, t_3)$ is a Bd-ideal of X. So, $x * y \in L(\mu, t_3)$. Thus, $\mu(x * y) \leq t_3 = \mu(x)$. Hence, $\mu^c(x * y) = 1 - \mu(x * y) \geq 1 - \mu(x) = \mu^c(x)$. Consequently, μ^c is a fuzzy Bd-ideal of X. □

Theorem 3.10. *Let μ be a fuzzy set of a Bd-algebra X. Then, μ^c is a fuzzy Bd-ideal of X if and only if, for every $t \in [0,1]$, $L^-(\mu, t) \neq \emptyset$ is a Bd-ideal of X.*

Proof. Assume that μ^c is a fuzzy Bd-ideal of X. Let $t \in [0,1]$ be such that $L^-(\mu, t) \neq \emptyset$. For every $x \in L^-(\mu, t)$, we have $\mu(x) < t$. Then, $1 - \mu(0) = \mu^c(0) \geq \mu^c(x) = 1 - \mu(x)$. So, $\mu(0) \leq \mu(x) < t$. Thus $0 \in L^-(\mu, t)$. Let $x * y \in L^-(\mu, t)$ and $y \in L^-(\mu, t)$. Also, $\mu(x * y) < t$ and $\mu(y) < t$. By the hypothesis, we have $1 - \mu(x) = \mu^c(x) \geq \min\{\mu^c(x * y), \mu^c(y)\} = \min\{1 - \mu(x * y), 1 - \mu(y)\} = 1 - \max\{\mu(x * y), \mu(y)\}$. It turns out that $\mu(x) \leq \max\{\mu(x * y), \mu(y)\} < t$ which implies that $x \in L^-(\mu, t)$. Next, for every $x \in L^-(\mu, t)$ and $y \in X$, we have $1 - \mu(x * y) = \mu^c(x * y) \geq \mu^c(x) = 1 - \mu(x)$. This means that $\mu(x * y) \leq \mu(x) < t$. Hence, $x * y \in L^-(\mu, t)$. Therefore, $L^-(\mu, t)$ is a Bd-ideal of X.

Conversely, assume that, for every $t \in [0,1]$, $L^-(\mu, t) \neq \emptyset$ is a Bd-ideal of X. Let $x, y \in X$. If $\mu^c(0) < \mu^c(x)$, then $1 - \mu(0) < 1 - \mu(x)$. Also, $\mu(0) > \mu(x)$. Thus, $x \in L^-(\mu, \mu(0))$. By assumption, we have $0 \in L^-(\mu, \mu(0))$. So, $\mu(0) < \mu(0)$, a contradiction. Hence, $\mu^c(0) \geq \mu^c(x)$. Suppose that $\mu^c(x) < \min\{\mu^c(x*y), \mu^c(y)\}$. Then $1 - \mu(x) < \min\{1 - \mu(x*y), 1 - \mu(y)\} = 1 - \max\{\mu(x*y), \mu(y)\}$. We have $\mu(x) > \max\{\mu(x*y), \mu(y)\}$. It follows that $x * y, y \in L^-(\mu, \mu(x)))$. By the hypothesis, we have $x \in L^-(\mu, \mu(x))$; that is, $\mu(x) < \mu(x)$, which is a contradiction. Hence, $\mu^c(x) \geq \min\{\mu^c(x * y), \mu^c(y)\}$. Similarly, suppose that $\mu^c(x*y) < \mu^c(x)$. Thus, $1 - \mu(x*y) < 1 - \mu(x)$. This implies that $\mu(x*y) > \mu(x)$. So, $x \in L^-(\mu, \mu(x*y))$. We obtain $L^-(\mu, \mu(x*y))$ is a Bd-ideal of X. This means that $x * y \in L^-(\mu, \mu(x * y))$. Then, $\mu(x * y) < \mu(x * y)$, a contradiction. Hence, $\mu^c(x * y) \geq \mu^c(x)$. Consequently, μ^c is a fuzzy Bd-ideal of X. □

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