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Induced Path Polynomials of the Join and Corona of Graphs

Cerina A. Villarta[1], Rolito G. Eballe[1], Rosalio G. Artes Jr.[2]

[1]Central Mindanao University
Bukidnon, Philippines

[2]Mindanao State University
Tawi-Tawi College of Technology and Oceanography
Tawi-Tawi, Philippines

email: cerina.villarta@cmu.edu.ph, rgaeballe@cmu.edu.ph,
rosalioartes@msutawi-tawi.edu.ph

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Abstract

In this paper, we establish the induced path polynomials of graphs resulting from the join and corona of two connected graphs.

1 Introduction

Graph polynomials captured a lot of attention in recent years because of their applications in Chemistry, Biology, and Physics [4]. Several discrete and applied mathematicians generated polynomials from graphs. Some interesting work had been done for star polynomials of graphs [1]. In our pioneering work in [5], we introduced the concept of induced path polynomials and showed that the induced path polynomial of a path is a linear combination of an n^{th} partial sum of a geometric series and its first derivative. Recently, in [2], another graph polynomial has been studied by considering geodetic closures of pairs of vertices in a graph.

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An *induced path* in G is a path induced by a subset of $V(G)$. The *induced path polynomial* of G is given by $P(G;x) = \sum_{i=1}^{\rho(G)} p_i(G)x^i$, where $p_i(G)$ is the number of induced paths in G of order i and $\rho(G)$ is the order of a maximum induced path in G.

For graph-theoretic concepts, the readers may refer to [3].

2 Join of Graphs

The following result characterizes the induced paths in graphs resulting from the join of two connected graphs.

Lemma 2.1. *Let $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. A subset $S = S_G \cup S_H$ induces a path in $G + H$ if and only if S satisfies one of the following conditions:*

(*i*) *S_G induces a path in G and $S_H = \varnothing$.*

(*ii*) *S_H induces a path in H and $S_G = \varnothing$.*

(*iii*) *S_G induces a P_2 in G' and S_H is a singleton set.*

(*iv*) *S_H induces a P_2 in H' and S_G is a singleton set.*

(*v*) *$|S_G| = |S_H| = 1$.*

Proof: Assume that S induces a path in $G + H$. Suppose that S does not satisfy $(i), (ii), (iii)$, and (iv). Then S_G and S_H are nonempty. If $\langle S_G \rangle$ or $\langle S_H \rangle$ contains an edge, then $\langle S \rangle$ contains a triangle. Thus $|E(\langle S_G \rangle)| = |E(\langle S_H \rangle)| = 0$. If $|S_G| = |S_H| = 2$, then S induces a C_4 in $G + H$. Moreover, if $|S_G| > 2$ or $|S_H| > 2$, then $\langle S \rangle$ contains a star S_3. Consequently, $|S_G| = |S_H| = 1$. The converse is clear. □

The induced path polynomial of the join of two graphs is established in the following theorem.

Theorem 2.2. *Let G and H be finite, simple and undirected graphs. Then*

$$\begin{aligned} P(G + H, x) &= P(G, x) + P(H, x) + |V(G)||V(H)|x^2 \\ &\quad +(|V(H)||E(G')| + |V(G)||E(H')|)x^3. \end{aligned}$$

Proof: From Lemma 2.1, for (i), we have $P(G,x)$. For (ii), we have $P(H,x)$. Condition (v) of the above lemma contributes $|V(G)||V(H)|x^2$ to the induced path polynomial representation of $G+H$. Moreover, a 2-subset of $V(G)$ induces a P_2 in G' whenever it generates an edge in G'. Similarly, a 2-subset of $V(H)$ induces a P_2 in H' whenever it generates an edge in H'. Thus conditions (iii) and (iv) of Lemma 2.1 contribute $(|V(H)||E(G')| + |V(G)||E(H')|)x^3$ to the induced path polynomial representation of $G+H$. Combining the terms gives the desired result. $\square$

Since $F_n = K_1 + P_n$, $W_n = K_1 + C_n$, and $K_{1,n} = K_1 + K_n'$, the following are direct consequences of Theorem 2.2.

Corollary 2.3. *For $n \geq 3$,*

(i) $P(F_n, x) = P(P_n, x) + x + nx^2 + \dfrac{(n-1)(n-2)}{2}x^3.$

(ii) $P(W_n, x) = P(C_n, x) + x + nx^2 + \dfrac{n(n-3)}{2}x^3.$

3 Corona of Graphs

The following result characterizes the induced paths in graphs resulting from the corona of two connected graphs.

Lemma 3.1. *Let $S_G \subseteq V(G)$ and for every $u, v \in V(G)$, where $u \neq v$ and let $S_{H_u} \subseteq V(H_u)$ and $S_{H_v} \subseteq V(H_v)$. A subset $S = S_G \cup S_{H_u} \cup S_{H_v}$ induces a path in $G \circ H$ if and only if S satisfies one of the following conditions:*

(i) *S_G induces a path in G and $|S| = |S_G|$.*

(ii) *S_G induces a path in G and $|S| = |S_G| + 1$.*

(iii) *S_G induces a path in G and $|S_{H_u}| = |S_{H_v}| = 1$.*

(iv) *$S_G = S_{H_v} = \varnothing$ and S_{H_u} induces a path in H_u.*

(v) *$|S_G| = 1$ and S_{H_v} induces a P_2 in H_v'.*

Proof: Assume that S induces a path in G. Suppose $(i), (ii), (iii)$, and (iv) do not hold. Necessarily, S_G must induce a path in G. Note that a vertex in G and an edge in a copy of H constitute a triangle in $G \circ H$. Hence (v) follows. The converse is clear. $\square$

Finally, we have the following result on the induced path polynomial of graphs resulting from the corona of two connected graphs.

Theorem 3.2. *Let G and H be finite, simple and undirected graphs. Then*

$$\begin{aligned} P(G \circ H, x) &= P(G,x) + |V(G)|P(H,x) \\ &\quad + |V(G)||V(H)|x^2 + |V(G)||E(H')|x^3 \\ &\quad + [x|V(H)|(P(G,x) - |V(G)|x)][2 + |V(H)|x]. \end{aligned}$$

Proof: By Lemma 3.1, (i) corresponds to $P(G,x)$. For (ii), we have $|V(G)||V(H)|x^2$ and $2[P(G,x) - |V(G)|x]|V(H)|x$. The third part of Lemma 3.1 corresponds to $[P(G,x) - |V(G)|x]|V(H)|^2x^2$. Part (iv) corresponds to $|V(G)|P(H,x)$. Finally, for each edge ab in H', $[a,u,b]$ is a P_3 in $G \circ H$ for every $u \in V(G)$. This gives $|V(G)||E(H')|x^3$. Combining gives the desired result. □

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