International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 917–920

On the Exponential Diophantine Equation $3^x + 121^y = z^2$

Malavika N., R. Venkatraman

Department of Mathematics,
Faculty of Engineering and Technology
SRM Institute of Science and Technology, Vadapalani Campus
No. 1, Jawaharlal Nehru Salai, Vadapalani,
Chennai-600026, Tamilnadu, India

email: mn8293@srmist.edu.in, venkatrr1@srmist.edu.in

(Received January 3, 2024, Accepted February 10, 2024,
Published February 12, 2024)

Abstract

In this paper, we show that the Diophantine equation $3^x + 121^y = z^2$ has precisely two solutions in non-negative integers; namely, $(1, 0, 2)$ and $(5, 2, 122)$.

1 Introduction

In recent years, mathematicians have focused on Exponential Diophantine equations, particularly those in the form $a^x + b^y = z^2$, where $(a, b, x, y, z) \in \mathbb{Z}_+$. In 2012, Sroysang [3] conducted a study on the Diophantine equation $3^x + 5^y = z^2$, determining that it has the unique solution $(1, 0, 2)$ within the domain of non-negative integers (x, y, z). In 2013, Rabago [4] conclusively solved two Diophantine equations; namely, $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, where x, y, and z are non-negative integers by identifying two solutions for each equation; specifically, $(1, 0, 2), (4, 1, 10)$ and $(1, 0, 2), (2, 1, 10)$, respectively. In 2020, Asthana and Singh [1] tackled the Diophantine equation $3^x + 117^y = z^2$ revealing precisely four solutions within the set of

Key words and phrases: Integer solutions, exponential Diophantine equation.
AMS (MOS) Subject Classifications: 11D61.
The corresponding author is R. Venkatraman.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

non-negative integers: $(1,0,2),(3,1,12),(7,1,48)$, and $(7,2,126)$. In 2023, Nongluk Viriyapong and Chokchai Viriyapong [5] considered the Diophantine equation $255^x+323^y=z^2$, proving exactly the two solutions $(1,0,16),(1,0,18)$ for non-negative integers x,y,z. Most approaches employed in tackling these equations have relied on established principles in number theory, including Catalan's conjecture, solved by Mihăilescu [2] in 2004, as well as fundamental concepts such as divisibility, congruence, and unique factorization.

2 Prerequisites

In this section, we shall recall the Catalan's Conjecture from 1844, which was subsequently proved by Mihăilescu in 2004.

Theorem 2.1 (Mihăilescu's Theorem). *Catalan's conjecture is true. That is, the Diophantine equation* $a^x-b^y=1$ *has the unique solution* $(a,b,x,y)=(3,2,2,3)$, *where* a,b,x *and* y *are integers with* $\min\{a,b,x,y\}>1$.

Lemma 2.2. *[3] The exponential Diophantine equation* $3^x+1=z^2$ *has a unique solution* $(1,2)$ *for the non-negative integers* x *and* z.

Lemma 2.3. *The exponential Diophantine equation* $1+121^y=z^2$ *has no non-negative integer solutions.*

Proof. If $y=0$, then $z^2=2$ which is a contradiction. Now, we have $y\geq 1$. By Catalan's Conjecture, we have $y=1$. Thus $z^2=122$, which is impossible. This completes the proof. □

3 Main results

Theorem 3.1. *The exponential Diophantine equation* $3^x+121^y=z^2$ *has precisely two non-negative integer solutions* $(1,0,2)$ *and* $(5,2,122)$.

Proof. Let x,y and z be non-negative integers such that $3^x+121^y=z^2$. By Lemma 2.3, we have $x\geq 1$. We have three cases for y:
Case I: y is zero. By Lemma 2.2, we have $(x,y,z)=(1,0,2)$.
Case II: y is even. Say $y=2k$, for some $k\in\mathbb{N}$. Then $3^x=z^2-121^y=(z-121^k)(z+121^k)$. Let $3^u=z-121^k$ and $3^{x-u}=z+121^k$, $x>2u$. As a result, we obtain $3^u[3^{x-2u}-1]=2.121^k$. For $k=1$, $3^u[3^{x-2u}-1]=2.121=2.11^2$. Thus $u=0$ and $3^x=243=3^5$ or $x=5$. This indicates that $x=5,y=2$ and $z=122$. Therefore, $(x,y,z)=(5,2,122)$.

Case III: y is odd. Say $y = 2k+1$, for some $k \in \mathbb{N}$. We will split this case into two segments.
Part 1: The equation $3^x + 121^y = z^2$ becomes $3^x + 121^{2k+1} = z^2$ or $3^x + 121.121^{2k} = z^2$. So $3^x - 3600.121^{2k} = z^2 - 3721.121^{2k} = (z - 61.121^k)(z + 61.121^k)$.

$$\begin{aligned} z - 61.121^k &= 1 &(3.1)\\ z + 61.121^k &= 3^x - 3600.121^{2k} &(3.2)\end{aligned}$$

Subtracting Eq. (3.1) from Eq. (3.2), we get $121^k[3600.121^k + 122] = 3^x - 1$. When $k = 0$, we get $3^x = 3723$ which is not solvable. Thus there are no solutions in this part.
Part 2: Again $3^x + 121^y = z^2$ becomes $3^x + 121^{2k+1} = z^2$. So $3^x + (2209 - 2088)121^{2k} = z^2$ or $3^x - 2088.121^{2k} = z^2 - 2209.121^{2k}$. Hence $3^x - 2088.121^{2k} = (z - 47.121^k)(z + 47.121^k)$.

$$\begin{aligned} z - 47.121^k &= 1 &(3.3)\\ z + 47.121^k &= 3^x - 2088.121^{2k} &(3.4)\end{aligned}$$

Subtracting Eq. (3.3) from Eq. (3.4), we get $121^k[2088.121^k + 94] = 3^x - 1$. This yields $k = 0$ and $3^x = 2183$ which remains insoluble. Thus there are no solutions in this part. □

Corollary 3.2. *For the Diophantine equation $3^x + 121^y = 4u^2$, where x, y, and u are non-negative integers, the solutions (x, y, u) are precisely given by $(1, 0, 1)$ and $(5, 2, 61)$.*

Proof. Let x, y, and u be non-negative integers satisfying the equation $3^x + 121^y = 4u^2$. Put $z = 2u$. Substituting this into the equation, we get $3^x + 121^y = z^2$. By Theorem 3.1, we have the set of solutions $(x, y, z) \in \{(1, 0, 2), (5, 2, 122)\}$. Consequently, u must belong to the set $\{1, 61\}$. Therefore, the non-negative integer solutions (x, y, u) for the Diophantine equation $3^x + 121^y = 4u^2$ are precisely $(1, 0, 1)$ and $(5, 2, 61)$. □

Corollary 3.3. *No non-negative integer solutions exist for the Diophantine equation $3^x + 121^y = r^4$.*

Proof. Assume that x, y, and r are non-negative integers satisfying the equation $3^x + 121^y = r^4$. Put $z = r^2$. Therefore, $3^x + 121^y = z^2$. By Theorem 3.1, $(x, y, z) \in \{(1, 0, 2), (5, 2, 122)\}$. Consequently, $r^2 = z \in \{2, 122\}$. Since z represents the square of some integer and 2 and 122 are not squares of any integer, the Diophantine equation $3^x + 121^y = r^4$ has no solution in the non-negative integers.

□

Corollary 3.4. $(1, 0, 1)$ *is the unique solution for the Diophantine equation* $3^x + 121^y = 4s^4$*, where* x, y*, and* s *are non-negative integers.*

Proof. Let x, y, and s be non-negative integers satisfying the equation $3^x + 121^y = 4s^4$. Put $z = 2s^2$. Then $3^x + 121^y = z^2$. By Theorem 3.1, $(x, y, z) = (1, 0, 2)$. Consequently, $2s^2 = 2$ which implies that $s = 1$. Therefore, $(1, 0, 1)$ is the unique non-negative integer solution for the equation $3^x + 121^y = 4s^4$, where x, y, and s are non-negative integers. □

4 Conclusion

In this paper, we demonstrated that the Exponential Diophantine equation $3^x + 121^y = z^2$ has exactly two solutions within the set of non-negative integers. These solutions are $(1, 0, 2)$ and $(5, 2, 122)$.

References

[1] Asthana Shivangi, Madan Mohan Singh, On the Diophantine equation $3^x + 117^y = z^2$, Ganita, **70,** (2020), 43–47.

[2] P. Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, J. Reine Angew. Math., **572,** (2004), 167–195.

[3] B. Sroysang, On the Diophantine Equation $3^x + 5^y = z^2$, Int. J. Pure and Appl. Math., **81,** no. 4, (2012), 605–608.

[4] J. F. T. Rabago, On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, Int. J. of Math. and Scientific Computing, **3,** no. 1, (2013), 28–29.

[5] Viriyapong Nongluk, Chokchai Viriyapong, On the Diophantine equation $255^x + 323^y = z^2$, Int. J. Math. Comput. Sci., **18,** no. 3, (2023), 521–523.