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Counting Perfect Matchings in Chain Graphs with the Specific Colored Faces

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Abstract

In this paper, we study counting perfect matchings in linear chain graphs, focusing on identically colored and alternatingly colored odd faces, using recurrence relations. Our primary objective is to derive explicit formulas for the numbers of perfect matchings in linear chain graphs with identically colored odd faces. Furthermore, we establish a relationship between the numbers of perfect matchings in linear chain graphs with identically colored odd faces and strip snake chain graphs. This relationship provides us with an alternative way of validating the numbers of perfect matchings in linear chain graphs with the same colored odd faces.

1 Introduction

All graphs considered in this paper will be finite, simple, and undirected. Let G be a connected graph. A subgraph M of G is called a *matching* in

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G if M contains no adjacent edges and no isolated vertex. A matching M of G is called a *perfect matching* in G if $V(M) = V(G)$, where $V(M)$ and $V(G)$ are the sets of all vertices in M and G, respectively. The number of perfect matchings in G is represented by $\phi(G)$. Perfect matchings in graphs are one of the fundamental objects that have been extensively studied in the field of graph theory. Counting perfect matchings in graphs has garnered significant attention in graph theory due to its relevance to various counting problems. Many researchers have studied counting perfect matchings in graphs as follows: Okamoto, Uehara, and Uno [8] and Štefankovič, Vigoda, and Wilmes [10] introduced algorithms for counting perfect matchings in graphs. Other ways to determine the numbers of perfect matchings in graphs were shown in [5, 9]. The problem of approximately counting perfect matchings in graphs was studied in [4, 1, 3]. Dong, Yan, and Zhang [2] showed the lower bound for the numbers of perfect matchings in the line graph of a graph and characterized all connected graphs that give the sharp lower bound. The recursive formulas for the numbers of perfect matchings in graphs were found by Marandi, Nejah, and Behmaram in [7].

Let G be a connected plane graph. A *face* of G is an induced subgraph of G which is a cycle. A face is classified as *odd* if it has an odd size and *even* if it has an even size. Furthermore, an even face of a size divisible by four is called a *blue face*, while an even face of a size that leaves a remainder of two when divided by four is called a *red face*. Conversely, an odd face is referred to as *black* if its size has a remainder of one when divided by four, and *pink* if its size has a remainder of three when divided by four.

For each integer $i, 1 \leq i \leq n$, let F_i be a face with edge set $E(F_i) = \{e_{i,1}, e_{i,2}, \ldots, e_{i,m_i}\}$, where m_i is the size of F_i. A *chain graph* G_n is defined as a connected plane graph with n faces $F_1, F_2, \ldots, F_n$ and having $n-1$ shared edges, denoted by the edge e_{i,k_i} in F_i and the edge $e_{i+1,1}$ in F_{i+1}, for each $i, 1 \leq i \leq n-1$, where k_i is referred to as a *shared edge index* of a chain graph G_n. Consequently, a chain graph G_n has size $m_1+m_2+\cdots+m_n-n+1$.

The chain graph G_6 shown in figure 1 consists of the black faces F_1, F_2, followed by the blue face F_3, the red face F_4, and the blue faces F_5 and F_6. These faces have sizes $m_1 = 5, m_2 = 5, m_3 = 8, m_4 = 6, m_5 = 4$ and $m_6 = 8$, respectively. Additionally, G_6 has five shared edges, namely $e_{1,3} = e_{2,1}, e_{2,4} = e_{3,1}, e_{3,4} = e_{4,1}, e_{4,3} = e_{5,1}$, and $e_{5,3} = e_{6,1}$ with respect to the shared edge indices $k_1 = 3, k_2 = 4, k_3 = 4, k_4 = 3$ and $k_5 = 3$.

It is interesting that the study of chain graphs can be related to chemical molecules. The structural formula of a chemical molecule is represented in terms of graph theory by a molecular graph, also known as a chemical

G_6 :

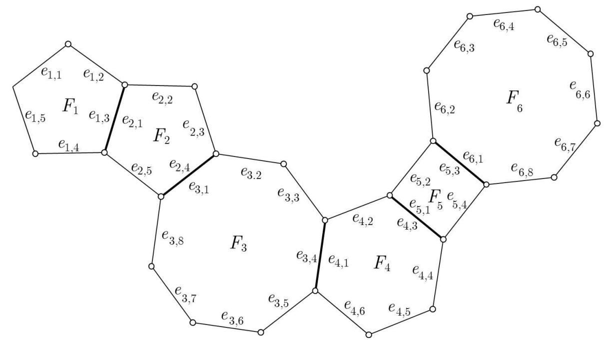

Figure 1: A chain graph G_6

graph in mathematical chemistry. Vertices in a chemical graph represent the compound's atoms, while edges represent chemical bonds. Chemical graphs are labeled graphs. Labels for the respective atom types are applied to its vertices, while labels for the bond types are applied to its edges.

Based on the findings discussed in [6], the authors employed a recurrence relation to count perfect matchings in various chain graphs G_n with n even faces, exclusively all the red faces, all the blue faces, and alternating faces. Motivated by these results, our study focuses on investigating the counting perfect matchings in chain graphs composed of all the odd faces.

2 Perfect Matchings of Linear Chain Graphs, Emphasizing Identically Colored and Alternatingly Colored Faces

Let n be a positive integer. An *linear chain graph* O_n is a chain graph with n odd faces of sizes $m_1, m_2, \ldots, m_n$, where a shared edge index $k_i = \lceil \frac{m_i}{2} \rceil + 1$ for each $i, 1 \leq i \leq n-1$.

O_4 :

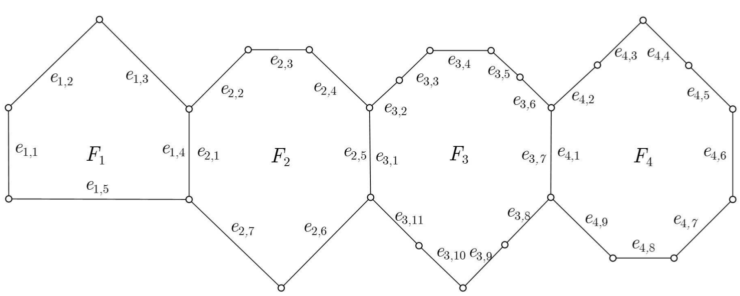

Figure 2: A linear chain graph O_4

In figure 2, the linear chain graph O_4 consists of the black face F_1, followed by the pink faces F_2, F_3, and the black face F_4 of sizes $m_1 = 5, m_2 = 7, m_3 = 11$, and $m_4 = 9$, respectively. O_4 contains three shared edges, $e_{1,4} = e_{2,1}, e_{2,5} = e_{3,1}$, and $e_{3,7} = e_{4,1}$ with respect to the shared edge indices $4, 5, 7$.

In this section, we determine the number of perfect matchings in a linear chain graph with identically colored and alternatingly colored odd faces using the recurrence relation. Since a linear chain graph of odd order does not have perfect matching, we only consider a linear chain graph with an even number of odd faces. The following result provides the recurrence relations for the number of perfect matchings in a linear chain graph containing all the black faces.

Theorem 2.1. *For every even positive integer n, let O_n be a linear chain graph with all the black faces, and d_n the number of perfect matchings in O_n. Then the recurrence relation $\phi(O_n) = d_n = 1 + d_{n-2}$, where n is even and $n \geq 4$ with the initial condition $d_2 = 2$.*

Proof. Let O_n be a linear chain graph consisting of black faces $D_1, D_2, \ldots, D_n$. Let M be a perfect matching in O_n and $E(M)$ the set of all edges in M.

For $n = 2$, we investigate the number of perfect matchings in O_2 considering two cases.

Case 1. $e_{1,1} \in E(M)$. Consider the black face D_1. Since $e_{1,1} \in E(M)$, $e_{1,2}, e_{1,m_1} \notin E(M)$. Then $E(M) \cap E(D_1) = \{e_{1,1}, e_{1,3}, \ldots, e_{1,k_1-1}\} \cup \{e_{1,m_1-1}, e_{1,m_1-3}, \ldots, e_{1,k_1+2}\}$. Consider the black face D_2. Since $e_{1,k_1-1}, e_{1,k_1+2} \in E(M)$, $e_{1,k_1} = e_{2,1}, e_{1,k_1+1} \notin E(M)$. Thus, $e_{2,m_2} \in E(M)$ and $e_{2,2} \notin E(M)$. Then $E(M) \cap E(D_2) = \{e_{2,3}, e_{2,5}, \ldots, e_{2,m_2}\}$. Therefore, O_2 has only one perfect matching M that contains $e_{1,1}$.

Case 2. $e_{1,1} \notin E(M)$. Consider the black face D_1. Since $e_{1,1} \notin E(M)$, $e_{1,2}, e_{1,m_1} \in E(M)$. Then $E(M) \cap E(D_1) = \{e_{1,2}, e_{1,4}, \ldots, e_{1,k_1-2}\} \cup \{e_{1,m_1}, e_{1,m_1-2}, \ldots, e_{1,k_1+1}\}$. Consider the black face D_2. Since $e_{1,k_1-2}, e_{1,k_1+1} \in E(M)$, $e_{1,k_1-1}, e_{1,k_1} \notin E(M)$. Thus, $e_{2,2} \in E(M)$ and $e_{2,m_2} \notin E(M)$. Then $E(M) \cap E(D_2) = \{e_{2,2}, e_{2,4}, \ldots, e_{2,m_2-1}\}$. Therefore, O_2 has only one perfect matching M that does not include $e_{1,1}$.

In both cases, the number of perfect matchings in O_2 consisting of the black faces D_1 and D_2 is $\phi(O_2) = d_2 = 1 + 1 = 2$.

Let $n \geq 4$. We consider the number of perfect matchings in O_n consisting of the black faces $D_1, D_2, \ldots, D_n$.

Case 1. $e_{1,1} \in E(M)$. Consider the black face D_1. Since $e_{1,1} \in E(M)$, $e_{1,2}, e_{1,m_1} \notin E(M)$. Then $E(M) \cap E(D_1) = \{e_{1,1}, e_{1,3}, \ldots, e_{1,k_1-1}\} \cup \{e_{1,m_1-1}, e_{1,m_1-3}, \ldots, e_{1,k_1+2}\}$. Consider the black face D_2. Since e_{1,k_1-1}, e_{1,k_1+2}

$\in E(M)$, $e_{1,k_1} = e_{2,1}, e_{1,k_1+1} \notin E(M)$. Thus, $e_{2,m_2} \in E(M)$ and $e_{2,2} \notin E(M)$. Then $E(M) \cap E(D_2) = \{e_{2,3}, e_{2,5}, \ldots, e_{2,m_2}\}$. We continue this process, following the same manner as in the previous step, until reaching step n. Consider the black face D_n. Since $e_{n-1,k_{n-1}-1}, e_{n-1,k_{n-1}+2} \in E(M)$, $e_{n-1,k_{n-1}} = e_{n,1}, e_{n-1,k_{n-1}+1} \notin E(M)$. Thus, $e_{n,m_n} \in E(M)$ and $e_{n,2} \notin E(M)$. Then $E(M) \cap E(D_n) = \{e_{n,3}, e_{n,5}, \ldots, e_{n,m_n}\}$. Therefore, there is only one perfect matching in O_n containing the edge $e_{1,1}$.

Case 2. $e_{1,1} \notin E(M)$. Consider the black face D_1. Since $e_{1,1} \notin E(M)$, $e_{1,2}, e_{1,m_1} \in E(M)$. Then $E(M) \cap E(D_1) = \{e_{1,2}, e_{1,4}, \ldots, e_{1,k_1-2}\} \cup \{e_{1,m_1}, e_{1,m_1-2}, \ldots, e_{1,k_1+1}\}$. Consider the black face D_2. Since $e_{1,k_1-2}, e_{1,k_1+1} \in E(M)$, $e_{1,k_1-1}, e_{1,k_1} \notin E(M)$. Thus, $e_{2,2} \in E(M)$ and $e_{2,m_2} \notin E(M)$. Since $e_{2,k_2-1}, e_{2,k_2+1} \notin E(M)$, either $e_{2,k_2} = e_{3,1} \in E(M)$ or $e_{2,k_2} = e_{3,1} \notin E(M)$. We will consider e_{2,k_2} in the next step. Then $E(M) \cap E(D_2 - e_{2,k_2}) = \{e_{2,2}, e_{2,4}, \ldots, e_{2,m_2-1}\} - \{e_{2,k_2}\}$. Consider the black face D_3. Since $e_{3,1} = e_{2,k_2}$ and $e_{2,k_2-1}, e_{2,k_2+1} \notin E(M)$, it is sufficient to consider the perfect matching M in O_{n-2} consisting of the black faces $D_3, D_4, \ldots, D_n$. That is we consider either $e_{3,1} \in E(M)$ or $e_{3,1} \notin E(M)$. Therefore, O_n has d_{n-2} perfect matchings that do not contain $e_{1,1}$.

In both cases, the number of perfect matchings in O_n consisting of the black faces $D_1, D_2, \ldots, D_n$ is $\phi(O_n) = d_n = 1 + d_{n-2}$.

Hence, we derive the recurrence relation $\phi(O_n) = d_n = 1 + d_{n-2}$, where n is even and $n \geq 4$ with the initial condition $d_2 = 2$. □

We now present the recurrence relations for the number of perfect matchings in a linear chain graph containing all the pink faces as follows:

Theorem 2.2. *For every even positive integer n, let O_n be a linear chain graph with all the pink faces and p_n the number of perfect matchings in O_n. Then the recurrence relation $\phi(O_n) = p_n = 1 + p_{n-2}$, where n is even and $n \geq 4$ with the initial condition $p_2 = 2$.*

With the aid of Theorems 2.1 and 2.2, we are able to establish the explicit formula for the number of perfect matchings in a linear chain graph with identically colored odd faces as follows:

Corollary 2.3. *For every even positive integer n, let O_n be a linear chain graph with all faces in the same color. Then $\phi(O_n) = \frac{n}{2} + 1$, where n is even and $n \geq 2$.*

Next, we present the recurrence relations for the numbers of perfect matchings in linear chain graphs with alternating colored faces of black and pink.

Theorem 2.4. *For every even positive integer n, let A_n be a linear chain graph consisting of alternating colored faces starting with the pink face and p_n the number of perfect matchings in A_n. Then the recurrence relation $\phi(A_n) = p_n = p_{n-2} + p_{n-4}$, where n is even and $n \geq 6$ with the initial conditions $p_2 = 2$ and $p_4 = 3$.*

Theorem 2.5. *For every even positive integer n, let A_n be a linear chain graph consisting of alternating colored faces starting with the black face and d_n the number of perfect matchings in A_n. Then the recurrence relation $\phi(A_n) = d_n = d_{n-2} + d_{n-4}$, where n is even and $n \geq 6$ with the initial conditions $d_2 = 2$ and $d_4 = 3$.*

3 The Relationship between Strip Snake Chain Graphs and Linear Chain Graphs

A *snake chain graph S_n* is a chain graph with n even faces of sizes $m_1, m_2, \ldots, m_n$, where the shared edge indices $k_1 = \frac{m_1}{2} + 1$ and $k_i \neq \lceil \frac{m_i}{2} \rceil + 1$ for some i, $2 \leq i \leq n-1$. The numbers of perfect matchings of snake chain graphs were studied in [6].

In particular types of snake chain graphs, we introduce the concept of a strip snake chain graph. For a snake chain graph S_n with n blue faces $B_1, B_2, \ldots, B_n$ where all shared edge indices are even, with the exception of the first shared edge, we define a *strip snake chain graph BS_n* obtained from a snake chain graph S_n by adding n new edges e_i $(1 \leq i \leq n)$ and joining two nonadjacent vertices in the blue faces B_i. Then, the blue face B_i with an edge e_i is called a blue strip face $B_i + e_i$ with a strip edge e_i of BS_n.

For instance, a strip snake chain graph BS_5 of figure 3 consists of the blue strip faces $B_1 + e_1, B_2 + e_2, B_3 + e_3, B_4 + e_4$, and $B_5 + e_5$ where shared edge indices $k_1 = 5, k_2 = 8, k_3 = 6$, and $k_4 = 6$.

In this section, we determine the number of perfect matchings of strip snake chain graphs. In order to do this, let us introduce some definitions and notation. For a strip snake chain graph BS_n with n blue strip faces $B_i + e_i$ where edge set $E(B_i) = \{e_{i,1}, e_{i,2}, \ldots, e_{i,m_i}\}$ for each $i, 1 \leq i \leq n$. First, we define the new edge set of B_i as $E(B_i) = \{f_{i,1}, f_{i,2}, \ldots, f_{i,m_i}\}$ by identifying share edge $e_{i,k_i} = f_{i,\frac{m_i}{2}+1}$ for each $i, 1 \leq i \leq n-1$ and $e_{n-1,k_{n-1}} = f_{n,1}$ and then arranging the remaining new edges in a clockwise manner.

To illustrate these concepts, consider a strip snake chain graph BS_5 of figure 3 having the new shared edges $f_{1,5} = e_{1,5} = e_{2,1}, f_{2,7} = e_{2,8} = e_{3,1}, f_{3,5} = e_{3,6} = e_{4,1}$, and $f_{4,5} = e_{4,6} = e_{5,1}$.

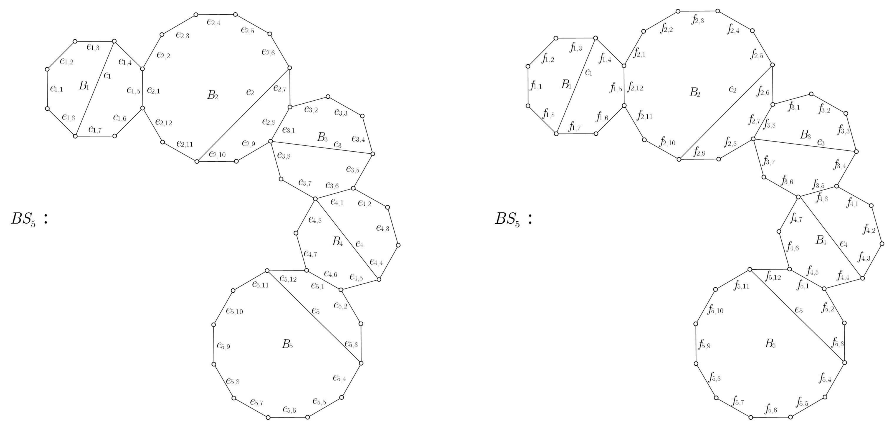

Figure 3: Strip snake chain graphs BS_5 consisting of the blue strip faces $B_1 + e_1, B_2 + e_2, B_3 + e_3, B_4 + e_4$, and $B_5 + e_5$

Now, we are able to present the recurrence relation for the numbers of perfect matchings in strip snake chain graphs as follows:

Theorem 3.1. *For every positive integer n, let BS_n be a strip snake chain graph and s_n the number of perfect matchings in BS_n. Then the recurrence relation $\phi(BS_n) = s_n = s_{n-1} + 1$, where $n \geq 2$ with the initial condition $s_1 = 2$.*

Proof. Let BS_n be a strip snake chain graph with the blue strip faces $B_1 + e_1, B_2 + e_2, \ldots, B_n + e_n$. For some $l_i, 3 \leq l_i \leq \frac{m_i}{2} - 1$, let $f_{i,l_i}, f_{i,l_i+1}, f_{i,m_i-l_i+2}$, and f_{i,m_i-l_i+3} be edges that are adjacent to the strip edge e_i. Let M be a perfect matching in BS_n and $E(M)$ the set of all edges in M.

Let $n = 1$. We consider the number of perfect matchings in BS_1 consisting of the blue strip face $B_1 + e_1$.

Case 1. $f_{1,1} \in E(M)$. Since $f_{1,1} \in E(M)$, $f_{1,2}, f_{1,m_1} \notin E(M)$. Then $\{f_{1,1}, f_{1,3}, \ldots, f_{1,m_1-1}\} \subseteq E(M)$. Since l_1 and $m_1 - l_1 + 3$ are not even or odd simultaneously, either $f_{1,l_1} \in E(M)$ or $f_{1,m_1-l_1+3} \in E(M)$. Thus, $e_1 \notin E(M)$. Then $E(M) = \{f_{1,1}, f_{1,3}, \ldots, f_{1,m_1-1}\}$. Hence, there exists exactly one perfect matching in BS_1 containing $f_{1,1}$.

Case 2. $f_{1,1} \notin E(M)$. Since $f_{1,1} \notin E(M)$, $f_{1,2}, f_{1,m_1} \in E(M)$. Then $\{f_{1,2}, f_{1,4}, \ldots, f_{1,m_1}\} \subseteq E(M)$. Since l_1 and $m_1 - l_1 + 3$ are not even or odd simultaneously, either $f_{1,l_1} \in E(M)$ or $f_{1,m_1-l_1+3} \in E(M)$. Thus, $e_1 \notin E(M)$. Then $E(M) = \{f_{1,2}, f_{1,4}, \ldots, f_{1,m_1}\}$. Hence, there exists only one perfect matching in BS_1 containing no $f_{1,1}$.

From *Case* 1 and *Case* 2, the number of perfect matchings in BS_1 consisting of the blue strip face $B_1 + e_1$ is $\phi(BS_1) = s_1 = 1 + 1 = 2$.

Let $n \geq 2$. We consider the number of perfect matchings in BS_n consisting of the blue strip faces $B_1 + e_1, B_2 + e_2, \ldots, B_n + e_n$.

Case 1. $f_{1,1} \in E(M)$. Consider the blue strip face $B_1 + e_1$. Since $f_{1,1} \in E(M)$, $f_{1,\frac{m_1}{2}}, f_{1,\frac{m_1}{2}+2} \notin E(M)$, and either the shared edge $f_{1,\frac{m_1}{2}+1} = e_{2,1} \in E(M)$ or $f_{1,\frac{m_1}{2}+1} = e_{2,1} \notin E(M)$. We will consider $f_{1,\frac{m_1}{2}+1}$ in the next step. Then $\{f_{1,1}, f_{1,3}, \ldots, f_{1,m_1-1}\} - \{f_{1,\frac{m_1}{2}+1}\} \subseteq E(M) \cap E(B_1 + e_1 - f_{1,\frac{m_1}{2}+1})$. Since l_1 and $m_1 - l_1 + 3$ are not even or odd simultaneously, either $f_{1,l_1} \in E(M)$ or $f_{1,m_1-l_1+3} \in E(M)$. Thus, $e_1 \notin E(M)$. Then $E(M) \cap E(B_1 + e_1 - f_{1,\frac{m_1}{2}+1}) = \{f_{1,1}, f_{1,3}, \ldots, f_{1,m_1-1}\} - \{f_{1,\frac{m_1}{2}+1}\}$. Consider the blue strip face $B_2 + e_2$. We have a shared edge $e_{2,1} = f_{1,\frac{m_1}{2}+1}$, and $f_{1,\frac{m_1}{2}}, f_{1,\frac{m_1}{2}+2} \notin E(M)$. If $e_{2,1} \in E(M)$, then $\{e_{2,1}, e_{2,3}, \ldots, e_{2,m_2-1}\} \subseteq E(M) \cap E(B_2 + e_2)$. Since k_2 is even, $e_{2,k_2} = f_{2,\frac{m_2}{2}+1} \notin E(M)$. Since $\frac{m_2}{2} + 1$ is odd, $f_{2,1} \notin E(M)$. If $e_{2,1} \notin E(M)$, then $\{e_{2,2}, e_{2,4}, \ldots, e_{2,m_2}\} \subseteq E(M) \cap E(B_2 + e_2)$. Since k_2 is even, $e_{2,k_2} = f_{2,\frac{m_2}{2}+1} \in E(M)$. Since $\frac{m_2}{2} + 1$ is odd, $f_{2,1} \in E(M)$. It is sufficient to consider the perfect matching M in BS_{n-1} consisting of the blue strip faces $B_2 + e_2, B_3 + e_3, \ldots, B_n + e_n$. That is, we consider either $f_{2,1} \in E(M)$ or $f_{2,1} \notin E(M)$. Hence, there exist s_{n-1} perfect matchings in BS_n containing $f_{1,1}$.

Case 2. $f_{1,1} \notin E(M)$. Consider the blue strip face $B_1 + e_1$. Since $f_{1,1} \notin E(M)$, $\{f_{1,2}, f_{1,4}, \ldots, f_{1,m_1}\} \subseteq E(M) \cap E(B_1 + e_1)$. Since l_1 and $m_1 - l_1 + 3$ are not even or odd simultaneously, either $f_{1,l_1} \in E(M)$ or $f_{1,m_1-l_1+3} \in E(M)$. Thus, $e_1 \notin E(M)$. Then $E(M) \cap E(B_1 + e_1) = \{f_{1,2}, f_{1,4}, \ldots, f_{1,m_1}\}$. Consider the blue strip face $B_2 + e_2$. Since $f_{1,\frac{m_1}{2}}, f_{1,\frac{m_1}{2}+2} \in E(M)$, $e_{2,1}, e_{2,2}, e_{2,m_2} \notin E(M)$. Then $\{e_{2,3}, e_{2,5}, \ldots, e_{2,m_2-1}\} \subseteq E(M) \cap E(B_2 + e_2)$. Since l_2 and $m_2 - l_2 + 3$ are not even or odd simultaneously and $l_2 + 1$ and $m_2 - l_2 + 2$ are not even or odd simultaneously, either $f_{2,l_2} \in E(M)$ or $f_{2,m_2-l_2+3} \in E(M)$ and either $f_{2,l_2+1} \in E(M)$ or $f_{2,m_2-l_2+2} \in E(M)$. Thus, $e_2 \notin E(M)$. Then $E(M) \cap E(B_2 + e_2) = \{e_{2,3}, e_{2,5}, \ldots, e_{2,m_2-1}\}$. Since k_2 is even, $e_{2,k_2} = f_{2,\frac{m_2}{2}+1} \notin E(M)$. Since $\frac{m_2}{2} + 1$ is odd, $f_{2,\frac{m_2}{2}}, f_{2,\frac{m_2}{2}+2} \in E(M)$. Proceed similarly up to step n. Consider the blue strip face $B_n + e_n$. Since $f_{n-1,\frac{m_{n-1}}{2}}, f_{n-1,\frac{m_{n-1}}{2}+2} \in E(M)$, $e_{n,1}, e_{n,2}, e_{n,m_n} \notin E(M)$. Then $\{e_{n,3}, e_{n,5}, \ldots, e_{n,m_n-1}\} \subseteq E(M) \cap E(B_n + e_n)$. Since l_n and $m_n - l_n + 3$ are not even or odd simultaneously and $l_n + 1$ and $m_n - l_n + 2$ are not even or odd simultaneously, either $f_{n,l_n} \in E(M)$ or $f_{n,m_n-l_n+3} \in E(M)$ and either $f_{n,l_n+1} \in E(M)$ or $f_{n,m_n-l_n+2} \in E(M)$. Thus, $e_n \notin E(M)$. Then $E(M) \cap E(B_n + e_n) = \{e_{n,3}, e_{n,5}, \ldots, e_{n,m_n-1}\}$. Hence, there exists exactly one perfect matching in BS_n containing no $f_{1,1}$ in BS_n.

From *Case* 1 and *Case* 2, the number of perfect matchings in BS_n consisting of the blue strip faces $B_1 + e_1, B_2 + e_2, \ldots, B_n + e_n$ is $\phi(BS_n) = s_n = s_{n-1} + 1$.

Therefore, we obtain the recurrence relation $\phi(BS_n) = s_n = s_{n-1} + 1$, where $n \geq 2$ with the initial condition $s_1 = 2$. $\square$

We obtain the explicit formula to determine the number of perfect matchings in a strip snake chain graph. This is achieved by using a recurrence relation as follows:

Corollary 3.2. *For every positive integer n, let BS_n be a strip snake chain graph. Then the number of perfect matchings is $\phi(BS_n) = n + 1$, where $n \geq 1$.*

The following corollary is an immediate consequence of the proof of Theorem 3.1.

Corollary 3.3. *A perfect matching in a strip snake chain graph contains no strip edge.*

In addition to counting perfect matchings in linear chain graphs with all faces in the same color using the recurrence relation as in the previous section, we now present the relationship between the numbers of perfect matchings in linear chain graphs with identically colored faces and strip snake chain graphs.

Theorem 3.4. *For every positive integer n, a linear chain graph with all faces in the same color O_{2n} is a strip snake chain graph BS_n. In particular, $\phi(O_{2n}) = \phi(BS_n)$.*

By combining Corollary 3.2 and Theorem 3.4, it allows us to use this relationship to verify the number of perfect matchings in a linear chain graph with all faces in the same color as the following theorem.

Theorem 3.5. *For every positive integer n, let O_{2n} be a linear chain graph with all faces in the same color. Then, $\phi(O_{2n}) = n + 1$.*

4 Conclusion

In this paper, we have discussed counting perfect matchings in linear chain graphs using recurrence relations with identically colored and alternatingly colored odd faces. We have obtained the explicit formulas for the numbers of perfect matchings in linear chain graphs with all faces in the same color, which depend on the number of their faces. Furthermore, the relationship between strip snake chain graphs and linear chain graphs provides us with an alternative way for validating the numbers of perfect matchings in linear chain graphs with the same colored odd faces.

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