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The Space of Prime Ideals of an Ordered Semigroup

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Abstract

The purpose of the present paper is to introduce the structure space of uniformly strongly prime ideals of an ordered semigroup and to study the separation axioms and the compactness property.

1 Introduction

By means of a closure operator, a topological space of the set of some types of ideals of an algebraic structure has been widely studied. A topological space of the set of primitive ideals of a ring was introduced and studied by Jacobson [7]. McCoy [9] observed that the set of generalized prime ideals of a ring can be treated in the same way. Kohls [8] introduced and investigated the space of prime ideals of a ring. Gillman [5] studied Hausdorff structure spaces of a ring. A topological space of several kinds of ideals of a semiring has been studied by Adhikari and Das [1]. A topological space of prime k-ideals of a Γ-semiring has also been studied by Goswami and Mukhopadhyay [6]. Chattopadhyay and Kar [2] considered the topological space of uniformly strongly prime ideals and of proper maximal ideals of a Γ-semigroup. The

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purpose of the present paper is to introduce the structure space of uniformly strongly prime ideals of an ordered semigroup and to study the separation axioms and the compactness property.

An *ordered semigroup* $(S, ., \leq)$ consists of a semigroup $(S, .)$ together with an ordered relation $\leq$ on S that is compatible with the semigroup operation; i.e., for any a, b, c in S, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$ [4]. For $A, B \subseteq S$, we write AB for $\{ab \in S \mid a \in A, b \in B\}$ and write $(A]$ for $\{x \in S \mid \exists a \in A, x \leq a\}$. It is observed that $A \subseteq (A]$; $A \subseteq B \Rightarrow (A] \subseteq (B]$; $((A]] = (A]$; $(A](B] \subseteq (AB]$; $((A](B]] = (AB]$; $(A \cup B] = (A] \cup (B]$; $(A \cap B] \subseteq (A] \cap (B]$. A non-empty subset A of S is called a *right ideal* (of S) if

(1) $ax \in A$ for any $a \in A$ and $x \in S$ (i.e., $AS \subseteq A$);

(2) $(A] = A$ (equivalently, if $a \in A$, $x \in S$ and $x \leq a$, then $x \in A$).

A left ideal of S can be defined similarly: a non-empty subset A of S is called a *left ideal* (of S) if

(1) $xa \in A$ for any $a \in A$ and $x \in S$ (i.e., $SA \subseteq A$);

(2) $(A] = A$ (equivalently, if $a \in A$, $x \in S$ and $x \leq a$, then $x \in A$).

A non-empty subset A of S is called a *two-sided ideal* (abbreviated by *ideal*) of S if it is both a left and a right ideal of S. If A and B are ideals of S, then $(AB]$ is an ideal of S. The principal ideal of S generated by an element a of S, denoted by $I(a)$, is of the form $I(a) = (a \cup aS \cup Sa \cup SaS]$ (i.e., $I(a) = (S^1aS^1]$). An element 0 of S is called a *zero* if $0a = a0 = 0$ for all $a \in S$. An element 1 of S is called an *identity* if $a1 = 1a = a$ for all $a \in S$.

Definition 1.1. *Let P be an ideal of an ordered semigroup $(S, ., \leq)$. Then P is called a* prime ideal *of S if for any ideals A and B of S, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

Theorem 1.2. *Let P be an ideal of an ordered semigroup $(S, ., \leq)$. Then P is a prime ideal of S if and only if for any $a, b \in S$, $aSb \subseteq P$ implies $a \in P$ or $b \in P$.*

Proof. Assume that P is a prime ideal of S and let $a, b \in S$ such that $aSb \subseteq P$. We have

$$\begin{aligned} ((S^1aS^1](S^1aS^1]]((S^1bS^1](S^1bS^1]] &\subseteq ((S^1aS^1)(S^1aS^1)]((S^1bS^1)(S^1bS^1)] \\ &\subseteq ((S^1aS^1)(S^1aS^1)(S^1bS^1)(S^1bS^1)] \\ &\subseteq (S^1aS^1aS^1bS^1bS^1] \\ &\subseteq P. \end{aligned}$$

By assumption, we have $((S^1aS^1](S^1aS^1]] \subseteq P$ or $((S^1bS^1](S^1bS^1]] \subseteq P$. Then $(S^1aS^1] \subseteq P$ or $(S^1bS^1] \subseteq P$ and so $a \in P$ or $b \in P$. Conversely, assume that for any $a, b \in S$, if $aSb \subseteq P$, then $a \in P$ or $b \in P$. Let A and B be ideals of S such that $AB \subseteq P$. To show that $A \subseteq P$ or $B \subseteq P$, we suppose $A \nsubseteq P$. Then there exists $a \in A \setminus P$. Let $b \in B$. We have

$$aSb \subseteq (aSb] \subseteq (ASB] \subseteq (AB] \subseteq (P] = P.$$

By assumption, we have $a \in P$ or $b \in P$. From $a \notin P$, we obtain $b \in P$. Thus $B \subseteq P$. Hence P is a prime ideal of S. □

Definition 1.3. *Let P be an ideal of an ordered semigroup $(S, ., \leq)$. Then P is said to be* uniformly strongly prime ideal *if S contains a finite subset F such that for any $x, y \in S$, $xFy \subseteq P$ implies $x \in P$ or $y \in P$.*

Theorem 1.4. *Let P be an ideal of an ordered semigroup $(S, ., \leq)$. If P is a uniformly strongly prime ideal of S, then P is a prime ideal of S.*

Proof. Suppose P is a uniformly strongly prime ideal of S. Then there exists a finite subset F of S such that for any $x, y \in S$, $xFy \subseteq P$ implies $x \in P$ or $y \in P$. Let $a, b \in S$ be such that $aSb \subseteq P$. Then $aFb \subseteq aSb \subseteq P$. So $a \in P$ or $b \in P$. Therefore, P is a prime ideal of S. □

2 The space of prime ideals

Let $\mathcal{A}$ denote the set of all uniformly strongly prime ideals of an ordered semigroup $(S, ., \leq)$ with zero. For $A \subseteq \mathcal{A}$, define

$$\bar{A} = \{I \in \mathcal{A} \mid \cap_{J \in A} J \subseteq I\}.$$

Observe that $\bar{\emptyset} = \emptyset$.

Theorem 2.1. *Let A, B be non-empty subsets of $\mathcal{A}$. Then the following hold:*

(1) $A \subseteq \bar{A}$;

(2) $\bar{\bar{A}} = \bar{A}$;

(3) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$;

(4) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof. (1): If $J_0 \in A$, then $\cap_{J\in A} J \subseteq J_0 y$. Therefore, $A \subseteq \bar{A}$.

(2): By (1), $\bar{A} \subseteq \bar{\bar{A}}$. Let $I \in \bar{\bar{A}}$. Then $\cap_{J\in\bar{A}} J \subseteq I$. From $\cap_{K\in A} K \subseteq J$ for all $J \in \bar{A}$, it follows that $\cap_{K\in A} K \subseteq \cap_{J\in\bar{A}} J \subseteq I$. So $I \in \bar{A}$. Hence $\bar{\bar{A}} \subseteq \bar{A}$. Consequently, $\bar{\bar{A}} = \bar{A}$.

(3): Assume that $A \subseteq B$. Let $I \in \bar{A}$. Then $\cap_{J\in A} J \subseteq I$. By assumption, $\cap_{K\in B} K \subseteq \cap_{J\in A} J \subseteq I$. So $I \in \bar{B}$. As a result, $\bar{A} \subseteq \bar{B}$.

(4): Since $A \subseteq A \cup B$, $\bar{A} \subseteq \overline{A \cup B}$. Similarly, since $B \subseteq A \cup B$, we have $\bar{B} \subseteq \overline{A \cup B}$. Then $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$. Now, let $I \in \overline{A \cup B}$. Then $\cap_{J\in A\cup B} J \subseteq I$. We have

$$\cap_{J\in A\cup B} J = \cap_{K\in A} K \cap \cap_{L\in B} L.$$

Since $\cap_{K\in A} K$ and $\cap_{L\in B} L$ are ideals of S, it follows that

$$\cap_{K\in A} K \cap_{L\in B} L \subseteq \cap_{K\in A} K \cap \cap_{L\in B} L = \cap_{J\in A\cup B} J \subseteq I.$$

Since I is prime (Theorem 1.4), $\cap_{K\in A} K \subseteq I$ or $\cap_{L\in B} L \subseteq I$. So $I \in \bar{A}$ or $I \in \bar{B}$. Hence $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$, and $\overline{A \cup B} = \bar{A} \cup \bar{B}$. □

From Theorem 2.1, we have the closure operator $A \to \bar{A}$ that gives a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$, called the *hull-kernel topology*. A pair $(\mathcal{A}; \tau_{\mathcal{A}})$ is called the *structure space* of the ordered semigroup $(S, ., \leq)$. For an ideal I of S, define

$$\Delta(I) = \{J \in \mathcal{A} \mid I \subseteq J\}, \;\; C\Delta(I) = \{J \in \mathcal{A} \mid I \nsubseteq J\}.$$

Theorem 2.2. *Any closed set in $\mathcal{A}$ is of the form $\Delta(I)$ for some ideal I.*

Proof. Let $\bar{A}$ be a closed set of the structure space $(\mathcal{A}; \tau_{\mathcal{A}})$ such that $A = \{I_\alpha \mid \alpha \in \Lambda\}$. We set $I = \cap_{\alpha\in\Lambda} I_\alpha$. Then I is an ideal of S. If $J \in \bar{A}$, then $I \subseteq J$ ands thus $J \in \Delta(I)$. If $J \in \Delta(I)$, then $I \subseteq J$. So $J \in \bar{A}$. Hence $\bar{A} = \Delta(I)$. □

Corollary 2.3. *Any open set in $\mathcal{A}$ is of the form $C\Delta(I)$ for some ideal I.*

For $a \in S$, let

$$\Delta(a) = \{J \in \mathcal{A} \mid a \in J\}, \;\; C\Delta(a) = \{J \in \mathcal{A} \mid a \notin J\}.$$

Theorem 2.4. *The set $\{C\Delta(a) \mid a \in S\}$ is an open base of $\tau_{\mathcal{A}}$.*

Proof. Let $U \in \tau_{\mathcal{A}}$ be an open set. Then $U = C\Delta(I)$ for some ideal I of S. Let $J \in U$. Then $I \nsubseteq J$ and so there exists $a \in S$ such that $a \in I \setminus J$. Thus $J \in C\Delta(a)$. Let $K \in C\Delta(a)$. Thus $a \notin K$. Since $I \nsubseteq K$, $K \in U$. Hence the set $\{C\Delta(a) \mid a \in S\}$ is an open base of $\tau_{\mathcal{A}}$. □

Theorem 2.5. *The structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the T_0-space.*

Proof. Let $I, J \in \mathcal{A}$ be two distinct elements. Then $I \nsubseteq J$ or $J \nsubseteq I$. If $I \nsubseteq J$, then $J \in C\Delta(I)$. Hence $C\Delta(I)$ is a neighbourhood of J not containing I. Therefore, the structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the T_0-space. □

Theorem 2.6. *The structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the T_1-space if and only if no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.*

Proof. Suppose that the structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the T_1-space. Let I' and J' be distinct elements of $\mathcal{A}$. Then $I' \in C\Delta(I)$ and $J' \in C\Delta(J)$ for some $I, J \in \mathcal{A}$ such that $J' \notin C\Delta(I)$ and $I' \notin C\Delta(J)$. If $I' \subseteq J'$, then $J \subseteq I' \subseteq J'$. This is a contradiction. Similarly, if $J' \subseteq I'$, then $I \subseteq J' \subseteq I'$. This is a contradiction. Hence no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.
Conversely, assume that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$. Let I' and J' be distinct elements of $\mathcal{A}$. Then $I' \nsubseteq J'$ and $J' \nsubseteq I'$. Then $I' \in C\Delta(J')$ and $J' \in C\Delta(I')$. We have that each of I' and J' has a neighborhood not containing the other. Hence the structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the T_1-space. □

Corollary 2.7. *If $\mathcal{M}$ is the set of all proper maximal ideals of an ordered semigroup S with identity, then $(\mathcal{M};\tau_{\mathcal{M}})$ is a T_1-space, where T_M is the induced topology on $\mathcal{M}$ from $(\mathcal{A};\tau_{\mathcal{A}})$.*

Theorem 2.8. *The structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the Hausdorff space if and only if for any distinct $I, J \in \mathcal{A}$ there exist $a, b \in S$ such that $a \notin I$ and $b \notin J$ and there does not exist $K \in \mathcal{A}$ such that $a \notin K$ and $b \notin K$.*

Proof. Suppose that the structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the Hausdorff space. Let I' and J' be distinct elements of $\mathcal{A}$. Then $I' \in C\Delta(I)$ and $J' \in C\Delta(J)$ for some $I, J \in \mathcal{A}$ and $C\Delta(I) \cap C\Delta(J) = \emptyset$. From $I' \in C\Delta(I)$, we have $I \nsubseteq I'$. Then there exists $a \in S$ such that $a \in I$ and $a \notin I'$. Similarly, from $J' \in C\Delta(J)$, there exists $b \in S$ such that $b \in J$ and $b \notin J'$. Assume that there exists $K \in \mathcal{A}$ such that $a \notin K$ and $b \notin K$. Then $K \in C\Delta(I)$ and $K \in C\Delta(J)$, so $C\Delta(I) \cap C\Delta(J) \neq \emptyset$. This is a contradiction. Therefore, there does not exist $K \in \mathcal{A}$ such that $a \notin K$ and $b \notin K$.
Conversely, assume that for any distinct $I, J \in \mathcal{A}$ there exist $a, b \in S$ such that $a \notin I$ and $b \notin J$ and there does not exist $K \in \mathcal{A}$ such that $a \notin K$ and $b \notin K$. Let I' and J' be distinct elements of $\mathcal{A}$; then there exist $a, b \in S$ such that $a \notin I'$ and $b \notin J'$ and there does not exist $K \in \mathcal{A}$ such that $a \notin K$ and $b \notin K$. We have $I' \in C\Delta(I(a))$ and $J' \in C\Delta(I(b))$. Since there does not exist $K \in \mathcal{A}$ such that $a \notin K$ and $b \notin K$, $C\Delta(I(a)) \cap C\Delta(I(b)) = \emptyset$. Hence the structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the Hausdorff space. □

Theorem 2.9. *The structure space* $(\mathcal{A};\tau_{\mathcal{A}})$ *is the regular space if and only if for any* $I\in\mathcal{A}$ *and* $a\in S\setminus I$ *there exists an ideal* J *of* S *such that* $I\in C\Delta(I(b))\subseteq\Delta(J)\subseteq C\Delta(I(a))$ *for some* $b\in S$.

Proof. Assume that the structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the regular space. Let $I\in\mathcal{A}$ and $a\in S\setminus I$. We have $I\in C\Delta(I(a))$ and $\mathcal{A}\setminus C\Delta(I(a))$ is a closed set. By assumption, there exist disjoint open sets U and V such that $I\in U$ and $\mathcal{A}\setminus C\Delta(I(a))\subseteq V$. We have $\mathcal{A}\setminus V\subseteq C\Delta(I(a))$ and $\mathcal{A}\setminus V$ is closed. By Theorem 2.2, $\mathcal{A}\setminus V=\Delta(J)$ for some ideal J of S. So $\Delta(J)\subseteq C\Delta(I(a))$. Since $U\cap V=\emptyset$, $V\subseteq\mathcal{A}\setminus U$. Since $\mathcal{A}\setminus U$ is closed, $\mathcal{A}\setminus U=\Delta(K)$ for some ideal K of S. We have $V\subseteq\Delta(K)$. Since $I\in U$, $I\notin\Delta(K)$. Then $K\nsubseteq I$. Therefore, there exists $b\in K\setminus I$ and so $I\in C\Delta(I(b))$. We claim that $V\subseteq\Delta(I(b))$. Let $L\in V$. Then $L\in\Delta(K)$. So $K\subseteq L$. Since $b\in K$, $b\in L$ and so $L\in\Delta(I(b))$. So the claim follows.
Since $\mathcal{A}\setminus\Delta(I(b))\subseteq\mathcal{A}\setminus V=\Delta(J)$, it follows that $C\Delta(I(b))\subseteq\Delta(J)$. Hence

$$I\in C\Delta(I(b))\subseteq\Delta(J)\subseteq C\Delta(I(a))$$

for some $b\in S$.
Conversely, assume that for any $I\in\mathcal{A}$ and $a\in S\setminus I$ there exists an ideal J of S such that $I\in C\Delta(I(b))\subseteq\Delta(J)\subseteq C\Delta(I(a))$ for some $b\in S$. Let $I\in\mathcal{A}$ and let $\Delta(K)$ be a closed set not containing I. Then $K\nsubseteq I$. So there exists $a\in S$ such that $a\in K$ and $a\notin I$. By assumption, there exists an ideal J of S such that

$$I\in C\Delta(I(b))\subseteq\Delta(J)\subseteq C\Delta(I(a))$$

for some $b\in S$. Since $a\in K$, $C\Delta(I(a))\cap\Delta(K)=\emptyset$. So

$$\Delta(K)\subseteq\mathcal{A}\setminus C\Delta(I(a))\subseteq\mathcal{A}\setminus\Delta(J).$$

We have $\mathcal{A}\setminus\Delta(J)$ is an open set with $C\Delta(I(b))\cap(\mathcal{A}\setminus\Delta(J))=\emptyset$. Therefore, the structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the regular space. □

Theorem 2.10. *The structure space* $(\mathcal{A};\tau_{\mathcal{A}})$ *is the compact space if and only if for any* $\{a_\alpha\in S\mid\alpha\in\Lambda\}$ *there exists a subset* $\{a_{\alpha_k}\mid k=1,2,\ldots,n\}$ *such that for any* $I\in\mathcal{A}$ *there exists* a_{α_k} *such that* $a_{\alpha_k}\notin I$.

Proof. Assume that the structure space $(\mathcal{A};\tau_{\mathcal{A}})$ is the compact space. We have that the set $\{C\Delta(I(a_\alpha))\mid a_\alpha\in S\}$ is an open cover of $\mathcal{A}$. Then there exists a finite set $\{C\Delta(I(a_i))\mid i=1,2,\ldots,n\}$ that is an open cover of $\mathcal{A}$. Hence $\{a_i\mid i=1,2,\ldots,n\}$ is a finite subcollection of elements of S such that for any $I\in\mathcal{A}$ there exists a_i such that $a_i\notin I$.

Conversely, assume that for any $\{a_\alpha \in S \mid \alpha \in \Lambda\}$ there exists a subset $\{a_{\alpha_k} \mid k = 1, 2, \ldots, n\}$ such that for any $I \in \mathcal{A}$ there exists a_{α_k} such that $a_{\alpha_k} \notin I$. Let $\{C\Delta(I(a_\alpha)) \mid a_\alpha \in S\}$ be an open cover of $\mathcal{A}$. Then there exists a finite set $\{a_{\alpha_k} \mid k = 1, 2, \ldots, n\}$ such that for any $I \in \mathcal{A}$ there exists a_{α_k} such that $a_{\alpha_k} \notin I$. We have $\{C\Delta(I(a_{\alpha_k})) \mid k = 1, 2, \ldots, n\}$ covers $\mathcal{A}$. Hence the structure space $(\mathcal{A}; \tau_{\mathcal{A}})$ is the compact space. □

Corollary 2.11. *If S is finitely generated, then the structure space $(\mathcal{A}; \tau_{\mathcal{A}})$ is the compact space.*

Proof. If $\{a_1, a_2, \ldots, a_n\}$ is a finite generating set of S, then for any $I \in \mathcal{A}$ there exist $a_{i_1}, a_{i_2}, \ldots, a_{i_k} \in S \setminus I$. By Theorem 2.10, the structure space $(\mathcal{A}; \tau_{\mathcal{A}})$ is the compact space. □

Let $(S, ., \leq)$ be an ordered semigroup. A element e in S is called an *idempotent* of S if $ee = e$. The set of all idempotents of S will be written as $E(S)$. An ideal I of S is said to be *full* if $E(S) \subseteq I$. Let $\mathcal{C}$ be the collection of all uniformly strongly prime full ideals of S. Then $\mathcal{C} \subseteq \mathcal{A}$ and so $(\mathcal{C}; \tau_{\mathcal{C}})$ is a subtopological space of $(\mathcal{A}; \tau_{\mathcal{A}})$.

Theorem 2.12. *The structure space $(\mathcal{C}; \tau_{\mathcal{C}})$ is the compact space.*

Proof. Let $\{\Delta(I_\alpha) \mid \alpha \in \Lambda\}$ be any collection of closed sets in $\mathcal{C}$ with the finite intersection property. Let I be the uniformly strongly prime full ideal generated by $E(S)$. Since any uniformly strongly prime full ideal J contains $E(S)$, we have J contains I; hence $I \in \cap_{\alpha \in \Lambda} \Delta(I_\alpha) \neq \emptyset$. Consequently, $(\mathcal{C}; \tau_{\mathcal{C}})$ is the compact space. □

Theorem 2.13. *The structure space $(\mathcal{C}; \tau_{\mathcal{C}})$ is the connected space.*

Proof. Let I be the uniformly strongly prime ideal generated by $E(S)$. Since any uniformly strongly prime full ideal J contains $E(S)$, we have J contains I. Hence I belongs to any closed set $\Delta(I')$ of $\mathcal{C}$. Therefore, any two closed sets of $\mathcal{C}$ are not disjoint. Consequently, $(\mathcal{C}; \tau_{\mathcal{C}})$ is the connected space. □

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