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On the Diophantine Equation $F_n^x + F_{n+1}^x = y^2$

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Abstract

We find all nonnegative integer solutions (n, x, y) to the Diophantine equation $F_n^x + F_{n+1}^x = y^2$, where F_n is the n-th Fibonacci number.

1 Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, where $F_0 = 0, F_1 = 1$ and $n \geq 0$. Numerous researchers have been investigating powers within the Fibonacci sequence, as documented in [1], [2], [3], [4], and [5]. A Diophantine equation is an equation in which only an integer solution is allowed. The Diophantine equation of the form

$$a^x + b^y = z^2,$$

where a and b are integers, has undergone extensive investigation by several researchers. However, the exploration of cases where a and b are the n-th Fibonacci numbers was pursued by Sroysang ([6], [7], [8]) and Acu [9]. In multiple papers, Sroysang demonstrated that the following Diophantine equation possesses solutions in nonnegative integers (x, y, z):

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- $2^x + 3^y = z^2$, with solutions of $(0, 1, 2)$, $(3, 0, 3)$, and $(4, 2, 5)$.
- $3^x + 5^y = z^2$, with solution of $(1, 0, 2)$.
- $8^x + 13^y = z^2$, with solution of $(1, 0, 3)$.

Additionally, Acu's study revealed that $2^x + 5^y = z^2$ has exactly two solutions in nonnegative integers $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$. Motivated by the aforementioned studies, we explore the nonnegative solutions (n, x, y) for equation $F_n^x + F_{n+1}^x = y^2$ further.

2 Preliminaries

Before presenting the main results, we review some properties of Fibonacci numbers along with some known results.

Here are some properties or identities we will use in this paper.

1. $\gcd(F_n, F_{n+1}) = 1, \quad$ for all $n > 0$
2. $F_n^2 + F_{n+1}^2 = F_{2n+1}, \quad$ for all $n \geq 0$

The following theorem is a result of Cohn [1].

Theorem 2.1. *The only Fibonacci numbers F_n that are perfect squares are* 0, 1, 144*: that is, when $n = 0, 1, 2, 12$.*

The next theorem can be found in [2].

Theorem 2.2. *Let p be an odd prime, a, b, c, k integers with $\gcd(a, b) = 1$ and $k \geq 2$. If $a^p + b^p = c^k$, then $a + b = d^k$ or $p^{k-1}d^k$, for some integer d.*

The following result can be found in [3].

Theorem 2.3. *The only positive integer solutions (n, k, p, y) to the equation $F_n = 3^k y^p$ with $k > 0$ and $p \geq 2$ are $F_4 = 3 \cdot 1$ and $F_{12} = 3^2 \cdot 4^2$.*

Finally, the next theorem can be obtained from [10].

Theorem 2.4. *The equation $x^n + y^n = z^2$ has no nontrivial primitive solutions for $n \geq 4$.*

3 Main results

Theorem 3.1. *All the solutions of the Diophantine equation* $F_n^x + F_{n+1}^x = y^2$ *with* $n = 0$ *in nonnegative integers* (n, x, y) *are of the form* $(0, x, 1)$*, where* $x \in \mathbb{N}$.

Proof. Let $n = 0$. We have $0^x + 1^x = y^2$. The value of x cannot be zero and so we get the desired form. □

Theorem 3.2. *All the nonnegative solutions of the Diophantine equation* $F_n^x + F_{n+1}^x = y^2$ *are* $(n, x, y) \in \{(10, 1, 12), (2, 3, 3)\}$ *for all* $n > 0$.

Proof. Let $n \neq 0$. We consider five cases:

Case 1. $x = 0$. Since $n \neq 0$, we get $1 + 1 = 2 = y^2$ which does not have integer solutions.

Case 2. $x = 1$. This implies that $F_n + F_{n+1} = y^2$ or $F_{n+2} = y^2$. By Theorem (2.1), either $n = 0$ or $n = 10$. Thus, $(n, x, y) = (10, 1, 12)$.

Case 3. $x = 2$. We have $F_n^2 + F_{n+1}^2 = y^2$. Using property 2, we get $F_{2n+1} = y^2$. By Theorem (2.1), we have $n = 0$. This is impossible since $n \neq 0$.

Case 4. $x = 3$. We have $F_n^3 + F_{n+1}^3 = y^2$. By Theorem (2.2), we get either $F_n + F_{n+1} = d^2$ or $F_n + F_{n+1} = 3d^2$. We note that $F_n + F_{n+1} = d^2$ has the same result as Case 2. Meanwhile, using Theorem (2.3), the equation $F_n + F_{n+1} = F_{n+2} = 3d^2$ yields $n = 2$. Thus we have $(n, x, y) = (2, 3, 3)$.

Case 5. $x \geq 4$. Using Theorem (2.4), we are guaranteed that the equation $F_n^x + F_{n+1}^x = y^2$ has no solutions. □

4 Conclusion

In this paper, we have shown that the only nonnegative integer solutions to the Diophantine equation $F_n^x + F_{n+1}^x = y^2$ are

$$(n, x, y) \in \{(0, x, 1), (10, 1, 12), (2, 3, 3)\},$$

where $x \in \mathbb{N}$.

5 Open Problem

For further exploration, one may investigate the nonnegative integer solutions of the Diophantine equation $F_n^x + F_{n+1}^y = z^2$ for $x \neq y$.

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