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The Solutions of the Diophantine Equations $p^x + p^y = z^q$ and $p^x - p^y = z^q$

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Abstract

Let p and q be prime numbers. In this article, we show that all non-negative integer solutions of the Diophantine equation $p^x+p^y=z^q$ are $(p,q,x,y,z)=(2,q,qt+q-1,qt+q-1,2^{t+1})$, $(2^q-1,q,qt+1,qt,2(2^q-1)^t)$, $(2,2,2t+3,2t,3\cdot 2^t)$, where t is a non-negative integer. All non-negative integer solutions of the Diophantine equation $p^x-p^y=z^q$ are $(p,q,x,y,z)=(p,q,t,t,0)$, $(2,q,qt+1,qt,2^t)$, $(4v^2+1,2,2t+1,2t,2v(4v^2+1)^t)$, $(3,3,3t+2,3t,2\cdot 3^t)$, where t is a non-negative integer and v is a positive integer.

1 Introduction

In 2019, Burshtein [1] considered the Diophantine equations $p^x+p^y=z^2$ and $p^x-p^y=z^2$, where p is a prime number. Burshtein proved that all positive integer solutions of the Diophantine equation $p^x+p^y=z^2$ are $(p,x,y,z)=(2,2t+1,2t+1,2^{t+1}),(3,2t+1,2t,2\cdot 3^t),(2,2t+3,2t,3\cdot 2^t)$, where t is a positive integer. All positive integer solutions of the Diophantine equation $p^x-p^y=z^2$ are $(p,x,y,z)=(2,2t+1,2t,2^t),(4v^2+1,2t+1,2t,2v(4v^2+1)^t)$, where t and

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v are positive integers. In this paper, we generalize the Burshtein's results to obtain all non-negative integer solutions of two Diophantine equations $p^x + p^y = z^q$ and $p^x - p^y = z^q$, where p and q are prime numbers.

2 Preliminaries

Theorem 2.1. *[1] Let p be an odd prime number. Then all positive integer solutions of the Diophantine equation $p^x - p^y = z^2$ are $(p, x, y, z) = (4v^2 + 1, 2t+1, 2t, 2v(4v^2+1)^t)$, where t and v are positive integers.*

Theorem 2.2. *[2] (Mihăilescu's Theorem) The equation $a^x - b^y = 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$, where a, b, x and y are positive integers with* $\min\{a, b, x, y\} > 1$.

3 Main Results

Theorem 3.1. *Let p and q be prime numbers. Then all non-negative integer solutions of the Diophantine equation $p^x + p^y = z^q$ are $(p, q, x, y, z) = (2, q, qt + q - 1, qt + q - 1, 2^{t+1})$, $(2^q - 1, q, qt + 1, qt, 2(2^q - 1)^t)$, $(2, 2, 2t + 3, 2t, 3 \cdot 2^t)$, where t is a non-negative integer.*

Proof. Without loss of generality, we may assume that $x \geq y$. If $x = y$, then $2p^x = z^q$. Since $q \geq 2$, we have $2 \mid p$. Therefore, $p = 2$. This implies that $z = 2^{\frac{x+1}{q}}$. Then $x = qt + q - 1$, for some non-negative integer t. Hence $(p, q, x, y, z) = (2, q, qt + q - 1, qt + q - 1, 2^{t+1})$.

Next, we consider $x > y$. Then $p^y(p^{x-y}+1) = z^q$. Since $\gcd(p^y, p^{x-y}+1) = 1$, there exist positive integers m and n such $z = mn$ with $p^{x-y}+1 = m^q$ and $p^y = n^q$. Assume that $m = 1$. Then $p^{x-y} = 0$, a contradiction. Thus $m > 1$.

Case 1. $x-y = 1$. Then $p = m^q - 1 = (m-1)(m^{q-1} + m^{q-2} + \cdots + m + 1)$. Therefore, $m = 2$ and so $p = 2^q - 1$. It follows that $(2^q - 1)^y = n^q$ or $n = (2^q - 1)^{\frac{y}{q}}$. Thus $y = qt$ for some non-negative integer t. Hence $(p, q, x, y, z) = (2^q - 1, q, qt + 1, qt, 2(2^q - 1)^t)$.

Case 2. $x-y > 1$. Since $m^q - p^{x-y} = 1$, we get $(m, p, q, x-y) = (3, 2, 2, 3)$, by Theorem 2.2. Then $n = 2^{\frac{y}{q}}$. This implies that $y = qt$, for some non-negative integer t. Hence $(p, q, x, y, z) = (2, 2, 2t + 3, 2t, 3 \cdot 2^t)$. □

Theorem 3.2. *Let p and q be prime numbers. Then all non-negative integer solutions of the Diophantine equation $p^x - p^y = z^q$ are $(p, q, x, y, z) = (p, q, t, t, 0)$, $(2, q, qt+1, qt, 2^t)$, $(4v^2+1, 2, 2t+1, 2t, 2v(4v^2+1)^t)$, $(3, 3, 3t+2, 3t, 2 \cdot 3^t)$, where t is a non-negative integer and v is a positive integer.*

Proof. If $x = y$, then $z = 0$. Hence $(p, q, x, y, z) = (p, q, t, t, 0)$, for some non-negative integer t. Next, we consider $x > y$. Then $p^y(p^{x-y} - 1) = z^q$. Since $\gcd(p^y, p^{x-y} - 1) = 1$, there exist positive integers m and n such $z = mn$ with $p^{x-y} - 1 = m^q$ and $p^y = n^q$. If $m = 1$, then $p^{x-y} = 2$. Therefore, $p = 2$ and $x - y = 1$. It follows that $2^y = n^q$ or $n = 2^{\frac{y}{q}}$. Then $y = qt$, for some non-negative integer t. Hence $(p, q, x, y, z) = (2, q, qt+1, qt, 2^t)$. For $m > 1$, we consider the following cases:

Case 1. $x - y = 1$. Then $p = m^q + 1$ and so $p > 2$. Assume that $q > 2$. Since q is prime, we get q is odd. Therefore, $p = (m+1)(m^{q-1} - m^{q-2} + \cdots + 1)$. This implies that $m + 1 = p$ and $m^{q-1} - m^{q-2} + \cdots + 1 = 1$. Then $m = 1$, a contradiction. Thus $q = 2$. By Theorem 2.1, we obtain $(p, q, x, y, z) = (4v^2+1, 2, 2t+1, 2t, 2v(4v^2+1)^t)$, where t is a non-negative integer and v is a positive integer.

Case 2. $x - y > 1$. Since $p^{x-y} - m^q = 1$, we obtain $(p, m, x-y, q) = (3, 2, 2, 3)$, by Theorem 2.2. Then $3^y = n^3$ and so $n = 3^{\frac{y}{3}}$. Thus $y = 3t$, for some non-negative integer t. Hence $(p, q, x, y, z) = (3, 3, 3t+2, 3t, 2 \cdot 3^t)$. □

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