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Movable Restrained-Domination in the Corona of Graphs

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Abstract

Let G be a simple connected graph. A dominating set $S \subseteq V(G)$ of G is a restrained dominating set in G if for every $v \in V(G)\backslash S$ there exists $u \in V(G)\backslash S$ such that $uv \in E(G)$. A restrained dominating set S of G is a 1-movable restrained-dominating set of G if for every $v \in S$ either $S\backslash \{v\}$ is a restrained dominating set of G or there exists $u \in (V(G)\backslash S) \cap N(v)$ such that $(S\backslash \{v\}) \cup \{u\}$ is a restrained dominating set of G. The minimum cardinality of a 1-movable restrained-dominating set in G, denoted by $\gamma^1_{mr}(G)$, is the 1-movable restrained-domination number of G.

In this paper, the 1-movable restrained-dominating sets in the corona of graphs are characterized. Also, the 1-movable restrained-domination numbers of these graphs are determined.

1 Introduction

Let $G = (V, E)$ be a simple graph. The open neighborhood of a vertex v of G is defined as the set $N_G(v) = \{u \in V(G) | uv \in E(G)\}$, while the closed neighborhood of v in G is defined as $N_G[v] = N_G(v) \cup \{v\}$. Any

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vertex $u \in N_G(v)$ is called a neighbor of v. The open neighborhood of a set $S \subseteq V(G)$ is defined as $N_G(S) = \bigcup\limits_{v \in S} N_G(v)$, while the closed neighborhood of a set S is defined as $N_G[S] = N_G(S) \cup S$.

A set $S \subseteq V(G)$ is a dominating set of G if for every $v \in V(G) \backslash S$ there exists $u \in S$ such that $uv \in E(G)$. A dominating set $S \subseteq V(G)$ is a restrained dominating set of G if for every $v \in V(G) \backslash S$ there exists $u \in V(G) \backslash S$ such that $uv \in E(G)$. Consequently, a dominating set S of G is a restrained dominating set of G if either $S = V(G)$ or $\langle V(G) \backslash S \rangle$ has no isolated vertices. The minimum cardinality of a restrained dominating set in G, denoted by $\gamma_r(G)$, is the restrained domination number of G.

Let G be a connected graph. A restrained dominating set S of G is a 1-movable restrained-dominating set of G if for every $v \in S$ either $S \backslash \{v\}$ is a restrained dominating set of G or there exists $u \in (V(G) \backslash S) \cap N(v)$ such that $(S \backslash \{v\}) \cup \{u\}$ is a restrained-dominating set of G. The minimum cardinality of a 1-movable restrained-dominating set in G, denoted by $\gamma^1_{mr}(G)$, is the 1-movable restrained-domination number of G. Any 1-movable restrained-dominating set of cardinality $\gamma^1_{mr}(G)$ is referred to as γ^1_{mr}-set of G. Not all graphs hasve a movable restrained-dominating set.

2 Main results

The following remarks follow from the definitions.

Remark 2.1. *If G has a 1-movable restrained-dominating set, then*

$$\gamma(G) \leq \gamma_r(G) \leq \gamma^1_{mr}(G).$$

Remark 2.2. *For every graph G, the vertex set $V(G)$ of G is not a 1-movable restrained-dominating set of G.*

Remark 2.3. *If G has a 1-movable restrained-dominating set, then*

$$\gamma^1_{mr}(G) \in \{1, 2, 3, \cdots, n-2\}.$$

Remark 2.4. *If S is a restrained dominating set of G, then every leaf of G is in S.*

Theorem 2.5. *If a connected graph G has a 1-movable restrained dominating set, then G has no leaves.*

Proof. Suppose that G has a 1-movable restrained-dominating set, say S. Suppose G has a leaf, say w. Let u be the support of w. Then, by Remark 2.4, $w \in S$. Suppose that $u \in S$. Then $S \setminus \{w\}$ is not a restrained dominating set of G. Suppose that $u \notin S$. Then $(S \setminus \{w\})$ and $(S \setminus \{w\}) \cup \{u\}$ are not restrained dominating sets of G.

Therefore, in either case, S is not a 1-movable restrained dominating set of G, contrary to our assumption.

Therefore, G has no leaves.

Remark 2.6. *The converse of Theorem* 2.5 *is not necessarily true.*

Let G and H be graphs of order m and n, respectively. The corona of two graphs G and H, denoted by, $G \circ H$, is the graph obtained by taking one copy of G and m copies of H and then joining the ith vertex of G to every vertex in the ith copy of H. For every $v \in V(G)$, we denote by H^v the copy of H whose vertices are joined or attached to the vertex v of G. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$, where $v \in V(G)$.

Theorem 2.7. *Let G and H be non-trivial connected graphs. Then $S \subseteq V(G \circ H)$ is a restrained dominating set in $G \circ H$ if and only if $S = A \cup B \cup C \cup D$, where $A \subseteq V(G)$, $B = \bigcup\{B_v : v \in A$ and either $B_v = \varnothing$ or $B_v \neq \varnothing$ and $\langle V(H^v) \setminus B_v \rangle$ has no isolated vertices $\}$, $C = \bigcup\{E_w : w \notin A, N_G(w) \cap A \neq \varnothing$ and E_w is a dominating set in H^w with $N_G(w) \cap (V(G) \setminus S) \neq \varnothing$ whenever $E_w = V(H^w)\}$ and $D = \bigcup\{D_w : w \notin A, N_G(w) \cap A = \varnothing$ and D_w is a dominating set in $H^w\}$.*

Proof. Suppose that S is a restrained dominating set in $G \circ H$. Let $A = V(G) \cap S$ and let $v \in A$. Put $B_v = V(H^v) \cap S$. Suppose that $B_v \neq \varnothing$. Since S is a restrained dominating set in $G \circ H$, $\langle V(H^v) \setminus B_v \rangle$ has no isolated vertices.

Next, let $w \in V(G) \setminus A$. Consider the following cases:

Case 1: Suppose that $w \in N_G(A)$.
Let $E_w = V(H^w) \cap S$. Since S is a dominating set in $G \circ H$ and $w \notin S$, it follows that E_w is a dominating set in H^w. Suppose that $E_w = V(H^w)$. Then since S is a restrained dominating set in $G \circ H$, $N_G(w) \cap (V(G) \setminus A) \neq \varnothing$.

Case 2: Suppose that $w \notin N_G(A)$.
Let $D_w = V(H^w) \cap S$. Since S is a restrained dominating set in $G \circ H$ and $w \notin S$, it follows that D_w is a dominating set in H^w.

Now, let $B = \bigcup\{B_v : v \in A$ and either $B_v = \varnothing$ or $B_v \neq \varnothing$ and $\langle V(H^v) \setminus B_v \rangle$

has no isolated vertices }, $C = \bigcup\{E_w : w \notin A, N_G(w) \cap A \neq \varnothing$ and E_w is a dominating set in H^w with $N_G(w) \cap (V(G) \backslash S) \neq \varnothing$ whenever $E_w = V(H^w)\}$ and $D = \bigcup\{D_w : w \notin A, N_G(w) \cap A = \varnothing$ and D_w is a dominating set in $H^w\}$. Then $S = A \cup B \cup C \cup D$.

For the converse, suppose that $S = A \cup B \cup C \cup D$, where A, B, C and D are described above. If $S = V(G \circ H)$, then S is a dominating set in $G \circ H$. Suppose that $S \neq V(G \circ H)$. Then $V(G \circ H) \backslash S \neq \varnothing$. Let $x \in V(G \circ H) \backslash S$ and let $w \in V(G)$ such that $x \in V(w + H^w)$. If $w \in S$, then $xw \in E(G \circ H)$. If $w \notin S$, then $x \in S_w = V(H^w \cap S$, where $S_w(D_w$ or $E_w)$ is a dominating set in H^w. Thus $\exists\ z \in V(H^w) \backslash S_w$ such that $xz \in E(G \circ H)$. Therefore, S is a dominating set in $G \circ H$.

Now, suppose that $S = V(G \circ H)$. Then S is a restrained dominating set in $G \circ H$. Suppose that $S \neq V(G \circ H)$. Let $w \in V(G \circ H) \backslash S$ and let $v \in V(G)$ such that $w \in V(v + H^v)$. Consider the following cases:

Case 1: Suppose that $w = v$.
Then $w \notin S$. If $w \in N_G(A)$, then $E_w = E_v$ is a dominating set in H^w. If $E_w \neq V(H^w)$, then $\exists\ y \in V(H^w) \backslash E_w$ such that $wy \in E(G \circ H)$. If $E_w = V(H^w)$, then by assumption $w \in N_G(V(G) \backslash A)$. Suppose that $w \notin N_G(A)$. Since G is connected and non-trivial, $\exists\ x \in V(G) \backslash A$ such that $wx \in E(G)$. Thus $\exists\ x \in V(G \circ H) \backslash S$ such that $wx \in E(G \circ H)$.

Case 2: Suppose that $w \neq v$.
Then $w \in V(H^v) \backslash B_v$, $w \in V(H^v) \backslash D_v$ or $w \in V(H^v) \backslash E_v$. Suppose that $w \in V(H^v) \backslash B_v$. Then $\langle V(H^v) \backslash B_v \rangle$ has no isolated vertices. Thus, $\exists\ y \in V(H^v) \backslash B_v$ such that $wy \in E(G \circ H)$. If $w \in V(H^v) \backslash D_v$ or $w \in V(H^v) \backslash E_v$, then $v \notin A$ and $wv \in E(G \circ H)$.

Therefore, in any case, $\langle V(G \circ H) \backslash S \rangle$ has no isolated vertices; that is, S is a restrained dominating set in $G \circ H$. $\square$

Corollary 2.8. *Let G and H be non-trivial connected graphs of order n and m, respectively. Then $\gamma_r(G \circ H) \leq n\gamma(H)$.*

The next result gives an upper bound for the 1-movable restrained-dominating set in the corona of graphs.

Theorem 2.9. *Let G and H be non-trivial connected graphs such that H has a 1-movable restrained-dominating set. Then $\gamma^1_{mr}(G \circ H) \leq |V(G)|\, \gamma^1_{mr}(H)$.*

Proof. Let S be a minimum 1-movable restrained-dominating set in H. For each $v \in V(G)$, let $S_v \subseteq V(H^v)$ such that $\langle S_v \rangle \cong \langle S \rangle$. Let $C = \bigcup_{v \in V(G)} S_v$.

Then, by Theorem 2.7, C is a restrained dominating set in $G \circ H$. Let $u \in C$ and $w \in V(G)$ such that $u \in V(H^w)$. Then $u \in S_w$. Since S_w is a 1-movable restrained-dominating set in H^w, $S_w \setminus \{u\}$ or $((S_w \setminus \{u\}) \cup \{z\})$ is a restrained dominating set in H^w for some $z \in N_{H^w}(u) \cap (V(H^w) \setminus S_w)$. Thus, by Theorem 2.7,
$C \setminus \{u\} = (\bigcup_{v \in V(G) \setminus \{w\}} S_v) \cup (S_w \setminus \{u\})$
or

$$(C \setminus \{u\}) \cup \{z\} = (\bigcup_{v \in V(G) \setminus \{w\}} S_v) \cup ((S_w \setminus \{u\}) \cup \{z\})$$

is a restrained dominating set in $G \circ H$ for some $z \in N_{G \circ H}(u) \cap (V(G \circ H) \setminus C)$. Hence, C is a 1-movable restrained-dominating set in $G \circ H$. Therefore,

$$\gamma^1_{mr}(G \circ H) \leq |C| = \sum_{v \in V(G)} |S_v| = |V(G)| \, \gamma^1_{mr}(H).$$

$\square$

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