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All Solutions of the Diophantine Equation $25^x - 7^y = z^2$

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Abstract

In this work, we show that the Diophantine equation $25^x - 7^y = z^2$ has only two non-negative integer solutions. The solutions (x, y, z) are $(0, 0, 0)$ and $(2, 2, 24)$.

1 Introduction

Nowadays, finding solutions of the Diophantine equation $a^x - b^y = z^2$ is a famous topic in the field of exponential Diophantine equations. Many mathematicians gave the non-negative integer solutions of the Diophantine equation, where a and b are explicit positive integers. In 2020, Burshtein [1] gathered all positive integer solutions of the Diophantine equations $13^x - 5^y = z^2$ and $19^x - 5^y = z^2$. In 2023, Tadee [3] investigated the Diophantine equations $9^x - 3^y = z^2$ and $13^x - 7^y = z^2$. Thongnak, Kaewong and Chuayjan ([5], [6]) discovered all non-negative integer solutions of the Diophantine equations $5^x - 3^y = z^2$ and $11^x - 17^y = z^2$, respectively. Moreover, Tadee and Wannaphan [4] studied the Diophantine equations $(p + a)^x - p^y = z^2$ and $p^x - (p + a)^y = z^2$, where a is a positive integer and p is a prime number.

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In this article, we investigate all non-negative integer solutions of the Diophantine equation

$$25^x - 7^y = z^2. \tag{1.1}$$

In 2004, Mihăilescu [2] presented an important theorem, which will be used to prove our result.

Theorem 1.1. *[2] (Mihăilescu's Theorem) The equation* $a^x - b^y = 1$ *has the unique solution* $(a, b, x, y) = (3, 2, 2, 3)$*, where* a, b, x *and* y *are positive integers with* $\min\{a, b, x, y\} > 1$.

2 Main result

Theorem 2.1. *All non-negative integer solutions* (x, y, z) *of* (1.1) *are* $(0, 0, 0)$ *and* $(2, 2, 24)$.

Proof. We consider the four exclusive cases:

Case 1. $x = 0$ and $y = 0$. From (1.1), we have $(x, y, z) = (0, 0, 0)$.

Case 2. $x = 0$ and $y > 0$. From (1.1), we have $z^2 < 0$, a contradiction.

Case 3. $x > 0$ and $y = 0$. From (1.1), we get $25^x - z^2 = 1$. It is easy to show that $x > 1$ and $z > 1$. This is impossible by Theorem 1.1.

Case 4. $x > 0$ and $y > 0$. From (1.1), we have $(5^x - z)(5^x + z) = 7^y$. Then there exists a non-negative integer u such that $5^x - z = 7^u$ and $5^x + z = 7^{y-u}$. Thus $2 \cdot 5^x = 7^u(7^{y-2u} + 1)$. Since $\gcd(7, 2 \cdot 5^x) = 1$, we have $u = 0$ and $2 \cdot 5^x = 7^y + 1$. Then $y \neq 1$. Assume that $y > 2$. Then $x > 2$ and $2 \cdot 5^x - 50 = 7^y + 1 - 50$. This implies that $50(5^{x-2} - 1) = 49(7^{y-2} - 1)$. Let $m = x - 2$ and $n = y - 2$. Then $50(5^m - 1) = 49(7^n - 1)$. Since $\gcd(5, 49) = 1$ and $\gcd(49, 50) = 1$, we can conclude that $5|(7^n - 1)$ and $49|(5^m - 1)$, respectively. Since $ord_5 7 = 4$ and $ord_{49} 5 = 42$, we obtain that $4|n$ and $42|m$, respectively. Then $m = 42l$ for some positive integer l. This implies that $50(5^{42l} - 1) = 49(7^n - 1)$. Since $5^{42l} \equiv 1 \pmod{31}$ and $\gcd(31, 49) = 1$, we obtain $31|(7^n - 1)$. Since $ord_{31} 7 = 15$, we get $15|n$. Thus $60|n$ and so $n = 60s$ for some positive integer s. This implies that $50(5^m - 1) = 49(7^{60s} - 1)$. Then $125|50(5^m - 1)$ because $7^{60s} \equiv 1 \pmod{125}$. Therefore, $5|(5^m - 1)$, a contradiction. Thus $y = 2$. Hence $(x, y, z) = (2, 2, 24)$. □

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