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A Direct Theorem on Angular Approximation Using k-Mixed Modulus of Smoothness

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Abstract

In approximation theory, the "direct theorem" is a fundamental concept that relates to how well more straightforward functions or signals can approximate a function or signal from a chosen approximation space. The direct theorem typically provides bounds on the error or the quality of the approximation in terms of various parameters, such as the degree of approximation, the dimension of the approximation space, and the properties of the approximated functions. The specific form of the direct theorem can vary depending on the context and the type of approximation being considered (e.g., polynomial approximation, Fourier series approximation, spline approximation, etc.). The direct theorem aims to establish the mathematical framework for comprehending how well elements from a given space can approximate a function, which is crucial in various fields like numerical analysis, signal processing, and scientific computing. Finally, the

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direct theorem is a crucial result in approximation theory. It describes how elements from a given approximation space get closer and better at approximating a given function as the number of terms in the approximation sequence grows. It provides essential insights into the behavior of approximations and plays a central role in various mathematical and computational disciplines. In this article, we aim to prove the direct theorem by defining a new appropriate operator for functions $L_p\left([0,2\pi]^d\right)$ that uses our new k-mixed modulus of smoothness to estimate the best approximation degree and the new k-mixed difference. Also, to prove the direct theorem, we introduce the angular approximation in $L_p\left([0,2\pi]^d\right.$ to reach the most accurate estimate.

1 Introduction

Suppose $f\left(x_1,\ldots,x_i,\ldots,x_d\right)$ is a measurable function on $[0,2\pi]^d$ and 2π-periodic for each variable. Then

$$f\left(x_1,\ldots,x_i,\ldots,x_d\right)\in L_p,\quad 0<p<1,$$

The functional

$$\|f(x)\|_p=\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|f\left(x_1,x_2,\ldots,x_i,\ldots,x_d\right)\right|^p dx_1dx_2\ldots dx_i\ldots dx_d\right)^{\frac{1}{p}} \tag{1.1}$$

is finite.
Bakhvalov and Nikolskii [2, 5] first developed classes of functions with a dominant mixed modulus of smoothness by defining the space

$$\mathcal{G}_p^{k_1,k_2}\mathfrak{X}\left(\mathbb{R}^2\right)=\left\{\begin{array}{r}\hbar\in L_p\left(\mathbb{R}^2\right):\|\hbar\|_{\mathcal{G}_p^{k_1,k_2}\mathfrak{X}(\mathbb{R}^2)}=\|\hbar\|_{L_p(\mathbb{R}^2)}+\left\|\dfrac{\partial^{k_1}\hbar}{\partial\varkappa_1^{k_1}}\right\|_{L_p(\mathbb{R}^2)}\\ +\left\|\dfrac{\partial^{k_2}\hbar}{\partial\varkappa_2^{k_2}}\right\|_{L_p(\mathbb{R}^2)}+\left\|\dfrac{\partial^{k_1+k_2}\hbar}{\partial\varkappa_1^{k_1}\partial\varkappa_2^{k_2}}\right\|_{L_p(\mathbb{R}^2)}\end{array}\right\} \tag{1.2}$$

The integer and fractionally mixed modulus of smoothness in one and n dimensions have been the subject of several recent studies (for a selection of these works, see [1, 7]).

The following denotes our k-mixed modulus of smoothness for functions

$f(x_1, \ldots, x_i, \ldots, x_d) \in L_p\left([0, 2\pi]^d\right), 0 < p < 1,$

$$\overline{\omega_k}(f, \delta)_p = \sup_{|h_i| \leq \delta} di = 1 \left\| \overline{\Delta_h^k} f(x_1, x_2, \ldots, x_i, \ldots, x_d) \right\|_p, \quad \delta > 0, \text{ or}$$
$$\underline{\omega_k}(f, \delta)_p = \sup_{|h_i| \leq \delta}^{d} \|\underline{\|}\| \underline{\Delta_h^k} f(x_1, x_2, \ldots, x_i, \ldots, x_d) \|_p \tag{1.3}$$

our $k-$mixed difference is defined as follows:

$$\overline{\Delta_h^k} f(x) = \sup_{i_{1 \leq i \leq d}} \left\{ \Delta_{h_1}^{k_1} f(x_1, \ldots, x_i, \ldots, x_d), \ldots, \Delta_{h_i}^{k_i} f(x_1, \ldots, x_i, \ldots, x_d), \ldots, \Delta_{h_d}^{k_d} f(x_1, \ldots, x_i, \ldots, x_d) \right\}, \text{ or}$$
$$\underline{\Delta_{\ln}^k} f(x) = \inf_{i_{1 \leq i \leq d}} \left\{ \Delta_{h_1}^{k_1} f(x_1, \ldots, x_i, \ldots, x_d), \ldots, \Delta_{h_i}^{k_i} f(x_1, \ldots, x_i, \ldots, x_d), \ldots, \Delta_{h_d}^{k_d} f(x_1, \ldots, x_i, \ldots, x_d) \right\}, \tag{1.4}$$

where

$$k = (k_1, k_2, \ldots, k_i, \ldots, k_d),$$
$$h = (h_1, h_2, \ldots, h_i, \ldots, h_d),$$
$$x = (x_1, x_2, \ldots, x_i, \ldots, x_d).$$

Potapov [6] gave the best approximation in terms of the angle. He has used theorems about angular approximation in the L_{p^*}-space, $1 \leq p \leq \infty$. We define the best angular approximation in $L_p\left([0, 2\pi]^d\right.$ by the formula

$$Y_{\ell_1, \ldots, \ell_d}(f)_p = \inf_{T_i \in \mathfrak{M}_i(\ell_i)_{p, i=1, \ldots, d}} \|f - T_1 - \cdots - T_d\|_p, \quad \ell_1, \ldots, \ell_d \in \mathbb{N} \tag{1.5}$$

where the symbol $\mathfrak{M}_i(\ell_i)_p$ represents the set of all functions from $L_p\left([0, 2\pi]^d\right.$ that are trigonometric polynomials of maximal order ℓ_i with respect to x_i. Many researchers submitted the direct theory of approximation using the usual and fractional modulus of smoothness for functions of one variable [2, 3] and continuous multivariate functions [7, 3, 4, 8].

2 Notation and Auxiliary Results

From generalized Jackson kernels [8], we obtain

$$J_{\ell_i}^{k_i}(x_i) = \frac{\gamma_{\ell_i}^{k_i}}{\ell_i^{2k_i - 1}} \left(\frac{\sin \frac{\ell_i x_i}{2}}{\sin \frac{x_i}{2}} \right)^{2k_i}, \ell_i, k_i = 1, 2, \ldots \tag{2.6}$$

where $i = 1, \dots, d$ and

$$\overline{J_\ell}\left(x_1, x_2, \dots, x_i, \dots, x_d\right) = \prod_{i=1}^{d} J_{\ell_i}^{k_i}\left(x_i\right) \tag{2.7}$$

and

$$\gamma_{\ell_i}^{k_i} = \frac{1}{\pi} \int_{-\pi}^{\pi} J_{\ell_i}^{k_i}\left(x_i\right) dx_i = 1.$$

Lemma 2.1. *Suppose* $x = \{x_1, x_2, \dots, x_i, \dots x_d\} \in \mathbb{R}^d . \ell_i, k_i = 1, 2, \dots, i = 1, \dots, d.$
Then

$$\frac{2}{2k_1\left(\ell_1 - 1\right) + 1} \cdots \frac{2}{2k_d\left(\ell_d - 1\right) + 1} \sum_{j_1=0}^{2k_1(\ell_1-1)} \cdots \sum_{j_d=0}^{2k_d(\ell_d-1)} \prod_{i=1}^{d} J_{\ell_i}^{k_i}\left(x_i - t_{\ell_i, k_i}^{j_i}\right) = 1,$$

where

$$t_{\ell_i, k_i}^{j_i} = \frac{2\pi j_i}{2k_i\left(\ell_i - 1\right) + 1}, j = 0, 1, \dots, k_i\left(\ell_i - 1\right), i = 1, \dots, d \tag{2.8}$$

Proof. We can prove this lemma by using statement 1 in [8] and via a simple calculation that takes into account the formula for the sum of a geometric progression's terms. □

Lemma 2.2. *Suppose* $k_i = 1, 2, \dots$ *and* $0 < p < 1$ *be such that* $k_i p > 1$. *Then for* $\ell_i = 1, 2, \dots, j_i = 0, 1, \dots, i = 1, \dots, d,$ *we obtain*

$$\begin{aligned} \int_0^{\pi} \int_0^{\pi} \cdots \int_0^{\pi} \cdots \int_0^{\pi} \prod_{i=1}^{d} x_i^{j_i} \left| J_{\ell_i}^{k_i}\left(x_i\right) \right|^p dx_1 dx_2 \dots dx_i \dots dx_d \\ \leq c\left(p, k_i\right) \prod_{i=1}^{d} \left(\ell_i^{p - j_i - 1}\right) \end{aligned} \tag{2.9}$$

Proof. We can prove this lemma using statement 2 in [8] and for $j = 0$ are provided in [3].

Let $j_i = 1$. Then, keeping in mind that $\sin \frac{x_i}{2} \geq \frac{x_i}{\pi}$ for $0 \leq x_i \leq \pi, \sin x_i \leq x_i$ for $x_i \geq 0$, while $k_i p > 1$, from equation (2.9), we successively get

$$\int_0^{\pi} \int_0^{\pi} \cdots \int_0^{\pi} \cdots \int_0^{\pi} \prod_{i=1}^{d} x_i^{j_i} \left| J_{\ell_i}^{k_i}\left(x_i\right) \right|^p dx_1 dx_2 \dots dx_i \dots dx_d$$

$$= \int_0^\pi \int_0^\pi \dots \int_0^\pi \dots \int_0^\pi \prod_{i=1}^{d} x_i^{j_i} \left| \frac{\gamma_{\ell_i}^{k_i}}{\ell_i^{2k_i-1}} \left(\frac{\sin \frac{\ell_i \boldsymbol{x}_i}{2}}{\sin \frac{\boldsymbol{x}_i}{2}} \right)^{2k_i} \right|^p dx_1 dx_2 \dots dx_i \dots dx_d$$

$$\leq \frac{c(p,k_i)}{\ell_1^{2k_1p-p} \dots \ell_d^{2k_dp-p}} \left(\ell_1^{2k_1p} \dots \ell_d^{2k_dp} \int_0^{1/n} \dots \int_0^{1/n} x_1 \times \dots \times x_d \; dx_1 \times \dots \times dx_d \right.$$
$$\left. + \int_{1/n}^{\infty} \dots \int_{1/n}^{\infty} x_1^{1-2k_1p} \times \dots \times x_d^{1-2k_dp} dx_1 \times \dots \times dx_d \right)$$
$$\leq c(p,k_1)\, \ell_1^{p-2} \dots \ell_d^{p-2}.$$

Similarly, we can prove equations (2.9) when $j_i = 2, 3, \dots$ □

Lemma 2.3. *Suppose that the function $f(x_1, \dots, x_i, \dots, x_d) \in L_p\left([0, 2\pi]^d\right)$, $0 < p < 1$. Then for all nonnegative $\gamma_i, i = 1, \dots, d$, and $\delta > 0$, we have the inequality*

$$\overline{\omega_k}(f, \gamma_i \delta)_p = \omega_{k_l}\left(f(x_1, \dots, \gamma_l x_l, \dots, x_d), \delta\right)_p$$
$$\leq (1+\gamma_l)^{k_l}\, \omega_{k_l}\left(f(x_1, \dots, x_l, \dots, x_d), \delta\right)_p.$$

Proof.

$$\overline{\omega_k}(f, \gamma_i \delta)_p = \sup_{|h_i| \leq \gamma_i \delta}^{\overset{d}{i=1}} \left\| \overline{\Delta_{\mathrm{lk}}^k} f(x_1, \dots, x_i, \dots, x_d) \right\|_p, \delta > 0$$
$$= \sup_{|h_i| \leq \gamma_i \delta}^{\overset{d}{i=1}} \left\| \sup_{i=1}^{d} \left\{ \Delta_{h_1}^{k_1} f(x_1, \dots, x_i, \dots, x_d), \dots, \Delta_{h_i}^{k_i} f(x_1, \dots, x_i, \dots, x_d), \right.\right.$$
$$\left.\left. \dots, \Delta_{h_d}^{k_d} f(x_1, \dots, x_i, \dots, x_d) \right\} \right\|_p$$
$$= \sup_{|h_i| \leq \delta}^{\overset{d}{i=1}} \left\| \sup_{i=1}^{d} \left\{ \Delta_{\gamma_1 h_1}^{k_1} f(x_1, \dots, x_i, \dots, x_d), \dots, \Delta_{\gamma_i h_i}^{k_i} f(x_1, \dots, x_i, \dots, x_d), \right.\right.$$
$$\left.\left. \dots, \Delta_{\gamma_d h_d}^{k_d} f(x_1, \dots, x_i, \dots, x_d) \right\} \right\|_p.$$

Choose $\Delta_{\gamma_l h_l}^{k_l} f(x_1, \dots, x_l, \dots, x_d)$ such that

$$\Delta_{\gamma_l h_l}^{k_l} f(x_1, \dots, x_l, \dots, x_d)$$
$$= \sup_{i=1}^{d} \left\{ \Delta_{\gamma_1 h_1}^{k_1} f(x_1, \dots, x_i, \dots, x_d), \dots, \Delta_{\gamma_i h_i}^{k_i} f(x_1, \dots, x_i, \dots, x_d), \right.$$
$$\left. \dots, \Delta_{\gamma_d h_d}^{k_d} f(x_1, \dots, x_i, \dots, x_d) \right\}$$

$$\overline{\omega_k}\left(f,\gamma_i\delta\right)_p=\sup_{|h_i|\le\delta}^{d}\left\|=\Delta^{k_l}_{\gamma_l h_l}f\left(x_1,\ldots,x_l,\ldots,x_d\right)\right\|_p .$$

By using the identity

$$\Delta^{k_l}_{\gamma h_l}f(x)=\sum_{j_1=0}^{\gamma-1}\cdots\sum_{j_i=0}^{\gamma-1}\cdots\sum_{j_k=0}^{\gamma-1}\Delta^{k_l}_{h_l}f\Big(x_1,\ldots,x_l+\left(k_l-j_1\right)h_l+\left(k_l-j_2\right)h_l+\cdots$$
$$+\left(k_l-j_{k_l}\right)h_l,\ldots,x_d\Big),$$

we have the inequality

$$\omega_{k_l}\left(f\left(x_1,\ldots,nx_l,\ldots,x_d\right),\delta\right)_p\le n^{k_l}\omega_{k_l}\left(f\left(x_1,\ldots,x_l,\ldots,x_d\right),\delta\right)_p .$$

For other values of $\gamma_l>0$

$$\begin{aligned}\overline{\omega_k}\left(f,\gamma_i\delta\right)_p&=\omega_{k_l}\left(f\left(x_1,\ldots,\gamma_l x_l,\ldots,x_d\right),\delta\right)_p\\&\le\left(1+\gamma_l\right)^{k_l}\omega_{k_l}\left(f\left(x_1,\ldots,x_l,\ldots,x_d\right),\delta\right)_p .\end{aligned}$$

□

3 Main Theorem

Theorem 3.1. *Assume that the function* $f(x)\in L_p\left([0,2\pi]^d\right),0<p<1$. *Then we have*

$$Y_{\ell_1,\ldots,\ell_d}(f)_p\le C(p)\overline{\omega_k}\left(f,\left(\frac{1}{\ell_1}\cdots\frac{1}{\ell_d}\right)\right)_p,\quad \ell_1,\ldots,\ell_d=0,1,\ldots$$

Proof. Since the set of continuous functions of 2π-periodic for each variable is dense in $L_p\left([0,2\pi]^d,0<p<1\right.$, we can assume that $f(x)$ is a continuous function. We put $k_i=\left[\frac{d}{p}\right]+1$ ([.] stands for the integer part). For arbitrary real numbers a_i, we have the functions:

$$\begin{aligned}\mathcal{F}_{\ell_1;a_1}f(x)&=\mathcal{F}_{\ell_1;a_1}f\left(x_1,x_2,\ldots,x_i,\ldots,x_d\right)\\&=\frac{2}{2k_1\left(\ell_1-1\right)+1}\sum_{j_1=0}^{2k_1(\ell_1-1)}\sum_{v_1=1}^{k_1}(-1)^{v_1}\begin{pmatrix}k_1\\v_1\end{pmatrix}\\&\quad f\left(\mathrm{t}^{j_1}_{\ell_1;k_1}+\left(k_1-v_1\right)a_1,x_2,\ldots,x_i,\ldots,x_\mathrm{d}\right)J^{k_1}_{\ell_1}\left(x_1-\mathrm{t}^{j_1}_{\ell_1;k_1}-a_1\right),\end{aligned}$$

$$\mathcal{F}_{\ell_2;a_2}f(\mathrm{x})=\mathcal{F}_{\ell_2;a_2}f\left(x_1,x_2,\ldots,x_i,\ldots,x_d\right)$$

$$= \frac{2}{2k_2(\ell_2-1)+1} \sum_{j_2=0}^{2k_2(\ell_2-1)} \sum_{v_2=1}^{k_2} (-1)^{v_2} \binom{k_2}{v_2}$$
$$f\left(x_1, t_{\ell_2;k_2}^{j_2} + (k_2-v_2)\,a_2, x_3, \ldots, x_i, \ldots, x_{\mathrm{d}}\right) J_{\ell_2}^{k_2}\left(x_2 - \mathrm{t}_{\ell_2;k_2}^{j_2} - a_2\right),$$

$$\vdots$$

$$\mathcal{F}_{\ell_i;a_i} f(\mathrm{x}) = \mathcal{F}_{\ell_i;a_i} f\left(x_1, x_2, \ldots, x_i, \ldots, x_d\right)$$
$$= \frac{2}{2k_i(\ell_i-1)+1} \sum_{j_i=0}^{2k_i(\ell_i-1)} \sum_{v_i=1}^{k_i} (-1)^{v_i} \binom{k_i}{v_i}$$
$$f\left(x_1, x_2, \ldots, \mathrm{t}_{\ell_i;k_i}^{j_i} + (k_i-v_i)\,a_i, \ldots, x_{\mathrm{d}}\right) J_{\ell_i}^{k_i}\left(x_i - \mathrm{t}_{\ell_i k_i}^{j_i} - a_i\right),$$

$$\vdots$$

$$\mathcal{F}_{\ell_d;a_d} f(\mathrm{x}) = \mathcal{F}_{\ell_d;a_d} f\left(x_1, x_2, \ldots, x_i, \ldots, x_d\right)$$
$$= \frac{2}{2k_d(\ell_d-1)+1} \sum_{j_d=0}^{2k_d(\ell_d-1)} \sum_{v_d=1}^{k_d} (-1)^{v_d} \binom{k_d}{v_d}$$
$$f\left(x_1, x_2, \ldots, x_i, \ldots, \mathrm{t}_{\ell_d;k_d}^{j_d} + (k_d-v_d)\,a_d\right) J_{\ell_d}^{k_d}\left(x_d - \mathrm{t}_{\ell_d;k_d}^{j_{j_d}} - a_d\right).$$

Using Lemmas 2.1, 2.2, and 2.3 and 2π-periodicity of the function $f\left(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_{\mathrm{i}}, \ldots, \mathrm{x}_{\mathrm{d}}\right)$, we successively get for $\ell_i = 1, 2, \ldots$

$$Y_{k_1(\ell_1-1),\ldots,k_i(\ell_i-1),\ldots,k_d(\ell_d-1)}(f)_p^p$$
$$\leq \left\{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \|f - \mathcal{F}_{\ell_1;a_1}(f) - \mathcal{F}_{\ell_2;a_2}(f) - \cdots - \mathcal{F}_{\ell_{\mathrm{i}};a_{\mathrm{i}}}(f)\right.$$
$$\left. - \cdots \mathcal{F}_{\ell_d;a_d}(f)\|_p^p \cdot da_1 \ldots da_d\right\}$$
$$\leq \frac{C(p)}{\ell_1^p \cdot \ell_2^p \ldots \ell_i^p \ldots \ell_d^p} \sum_{j_1=0}^{k_1(\ell_1-1)+1} \sum_{j_2=0}^{k_2(\ell_2-1)+1} \cdots \sum_{j_i=0}^{k_i(\ell_{\mathrm{i}}-1)+1} \cdots \sum_{j_{\mathrm{d}}=0}^{k_d(\ell_{\mathrm{d}}-1)+1} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots$$
$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left\{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\right| f\left(x_1, x_2, \ldots, x_i, \ldots, x_{\mathrm{d}}\right)$$
$$- \sum_{v_1=1}^{k_1} (-1)^{v_1} \binom{k_1}{v_1} f\left(\mathrm{t}_{\ell_1;k_1}^{j_1} + (k_1-v_1)\,a_1, \ldots, \mathrm{x}_{\mathrm{i}}, \ldots, \mathrm{x}_{\mathrm{d}}\right) - \ldots$$
$$- \sum_{v_i=1}^{k_i} (-1)^{v_i} \binom{k_i}{v_i} f\left(x_1, \ldots, \mathrm{t}_{\ell_i;k_i}^{j_i} + (k_i-v_i)\,a_{\mathrm{i}}, \ldots, \mathrm{x}_{\mathrm{d}}\right) - \cdots$$

$$- \sum_{v_d=1}^{k_d} (-1)^{v_d} \binom{k_d}{v_d} f\left(x_1, \dots, x_i, \dots, \mathrm{t}_{\ell_d;k_d}^{j_d} + (k_d - v_d)\, \mathrm{a_d}\right)\Big|^p$$

$$\cdot \prod_{i=1}^{d} \Big(J_{\ell_i}^{k_i}\left(x_i - \mathrm{t}_{\ell_i;k_i}^{j_i} - \mathrm{a_i}\right)\Big)^p da_1 da_2 \dots da_i \dots da_d \Big\}. dx_1 dx_2 \dots dx_i \dots dx_\mathrm{d}$$

$$\leq C(p)\left(\ell_1 \dots \ell_d\right)^{1-p} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \Big\{ \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \Big| f\left(x_1, \dots, x_i, \dots, x_d\right)$$

$$- \sum_{v_1=1}^{k_1} (-1)^{v_1} \binom{k_1}{v_1} f\left((k_1 - v_1)\, a_1, \dots, x_i, \dots, \mathrm{x}_d\right) - \dots$$

$$- \sum_{v_i=1}^{k_i} (-1)^{v_i} \binom{k_i}{v_i} f\left(x_1, \dots, (v_i - k_i)\, a_i, \dots, \mathrm{x}_d\right) - \cdots$$

$$- f\left(x_1, \dots, x_i, \dots, (k_d - v_d)\, \mathrm{a}_d\right)\Big|^p \prod_{i=1}^{d} \left(J_{\ell_i}^{k_i}\left(x_i - a_\mathrm{i}\right)\right)^p da_1 \dots da_d \Big\}. dx_1 \dots dx_\mathrm{d}$$

$$\leq C(p)\left(\ell_1 \dots \ell_d\right)^{1-p} \cdot \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \left\| \Delta_h^k f\left(x_1, x_2, \dots, x_i, \dots, x_d\right) \right\|_p^p$$

$$\cdot \prod_{i=1}^{d} \left(J_{\ell_i}^{k_i}\left(h_i\right)\right)^p d\, \mathrm{h}_1 dh_2 \dots dh_i \dots d\, \mathrm{h}_d$$

$$Y_{k_1(\ell_1-1),\dots,k_i(\ell_i-1),\dots,k_d(\ell_d-1)}(f)_p^p \leq C(p)\left(\ell_1 \dots \ell_d\right)^{1-p} \overline{\omega_k}\left(f, \left(\frac{1}{\ell_1} \dots \frac{1}{\ell_d}\right)\right)_p^p$$

$$\cdot \prod_{i=1}^{\mathrm{d}} \int_{-\pi}^{\pi} \left(1 + \ell_i \left|h_i\right|\right) \left(J_{\ell_i}^{k_i}\left(h_i\right)\right)^p dh_i$$

$$\leq C(p) \overline{\omega_k}\left(f, \left(\frac{1}{\ell_1} \dots \frac{1}{\ell_d}\right)\right)_p^p.$$

□

4 Conclusion

In this study, we established a new suitable operator for functions in $L_p\left([0, 2\pi]^d\right)$. The angular degree of the best approximation of f can reach zero using $k-$mixed modulus of smoothness.

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