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Goldie Supplement Extending Modules

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Abstract

In this paper, we call the module $\mathcal{N}$ the Goldie supplement extending module ($\mathcal{G}SCS$-module) which is a new generalization weaker than the Goldie extending module and stronger than the Goldie weakly supplement extending module. We discuss the relation between Goldie supplement extending module and other notions such as the Goldie extending module and the supplement extending module. Moreover, we explain when the submodule of Goldie supplement extending is a Goldie supplement extending module. Furthermore, we show that the direct sum of $\mathcal{G}SCS$-module holds only under certain conditions.

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1 Introduction

Throughout this work, $\mathcal{N}$ is a unitary module over a commutative ring R with identity. A submodule U is essential [1] in $\mathcal{N}$ $(U \leq_e \mathcal{N})$, if $U \cap S \neq 0$ $\forall\ 0 \neq S \leq \mathcal{N}$. In addition, U is closed in $\mathcal{N}$ ($U \leq_c \mathcal{N}$), if $L \leq_e U \leq \mathcal{N}$, then $L = U$. A submodule L is rational in $\mathcal{N}$ $(L \leq_r \mathcal{N})$, if for each $a, b \in \mathcal{N}$ with $a \neq 0\ \exists\ r \in R \ni rb \in L$ and $ra \neq 0$. A singular submodule of $\mathcal{N}$ is $Z(\mathcal{N}) = \{a \in \mathcal{N} | La = 0;$ for some essential ideal L of $R\}$. If $Z(\mathcal{N}) = \mathcal{N}$, then $\mathcal{N}$ is a singular module and if $Z(\mathcal{N}) = 0$, then $\mathcal{N}$ is non-singular module. S is small [2] in $\mathcal{N}$ $(S \ll \mathcal{N})$, if whenever $S + F = \mathcal{N}$, then $F = \mathcal{N}$.

A submodule E is supplement [3] to a submodule D in $\mathcal{N}$ ($E \leq_s \mathcal{N}$), if $E + D = \mathcal{N}$ and $E \cap D \ll E$. A module $\mathcal{N}$ is called supplemented, when each submodule of $\mathcal{N}$ has a supplement in $\mathcal{N}$. A submodule W is weakly supplement of a submodule B in $\mathcal{N}$ ($E \leq_{ws} \mathcal{N}$), if $W + B = \mathcal{N}$ and $W \cap B \ll \mathcal{N}$. The Jacobson radical [1] of $\mathcal{N}$ $(Rad(\mathcal{N})) = \sum_{L>>\mathcal{N}} L$.

A module $\mathcal{N}$ is extending (denoted by CS-module) [4], if $\forall P \leq \mathcal{N} \Longrightarrow P \leq_e W \leq_\oplus \mathcal{N}$. For more details about the generalizations of extending module see [5, 6].

$\mathcal{N}$ is supplement extending module (denoted by SCS-module) [7], if $\forall F \leq \mathcal{N} \Longrightarrow F \leq_e H \leq_\oplus \mathcal{N}$.

In [8], the relation α and β between the submodules L and V of $\mathcal{N}$ was introduced . $L\ \alpha\ V$ if and only if there exists $E \leq \mathcal{N} \ni L \leq_e E$ and $V \leq_e E$, the relation α is reflexive and symmetric. $L\beta V$ if and only if $L \cap V \leq_e L$ and $L \cap V \leq_e V$, the relation β is reflexive, symmetric and transitive. A module $\mathcal{N}$ is called Goldie extending (denoted by $\mathcal{G}CS$-module), $\forall F \leq \mathcal{N}, \exists C \leq_\oplus \mathcal{N} \ni F\beta C$. Every CS-module is $\mathcal{G}CS$-module.

A module $\mathcal{N}$ is Goldie weakly supplement extending ($\mathcal{G}WSCS$-module) [9], if $\forall F \leq \mathcal{N}, \exists W \leq_{WS} \mathcal{N} \ni F\beta W$. Every $\mathcal{G}CS$-module is $\mathcal{G}WSCS$-module.

In [10], the relations α_r and β_r between the submodules L and V of $\mathcal{N}$ was introduced. $L\ \alpha_r V$ if and only if there is $E \leq \mathcal{N} \ni L \leq_r E$ and $V \leq_r E$. $L\ \beta_r\ V$ if and only if $L \cap V \leq_r L$ and $L \cap V \leq_r V$. $\mathcal{N}$ is called Goldie rationally extending (denoted by $\mathcal{G}RCS$-module), if $\forall H \leq \mathcal{N}, \exists S \leq_\oplus \mathcal{N} \ni H\beta_r S$. A module $\mathcal{N}$ is called Goldie supplement rationally extending (denoted by $\mathcal{G}SRCS$-module) [5], if $\forall E \leq \mathcal{N}, \exists S \leq_S \mathcal{N} \ni E\beta_r S$. Every $\mathcal{G}RCS$-module is $\mathcal{G}SRCS$-module.

2 Goldie supplement extending modules

Definition 2.1. *$\mathcal{N}$ is called Goldie supplement extending module denoted by $\mathcal{G}SCS$-module, if every $Y \leq \mathcal{N}$ there is a supplement $D \leq \mathcal{N} \ni Y\beta D$.*

Proposition 2.2. *$\mathcal{N}$ is called $\mathcal{G}SCS$-module if and only if $\forall W \leq_c \mathcal{N}$ $\exists$ a $P \leq_s \mathcal{N} \ni W\beta P$.*

Proof. Let $W \leq_c \mathcal{N}$ and $\mathcal{N}$ be a $\mathcal{G}SCS$-module. Then there is a $F \leq_s \mathcal{N}$ $\ni W\beta F$. By contrast, let $C \leq \mathcal{N}$. Then there exists $W \leq_c \mathcal{N} \ni C \leq_e W$ and by hypothesis there is a $F \leq_s \mathcal{N} \ni W\beta F$. Since $C = C \cap W \leq_e W$ and $C \leq_e C$, $C\ \beta W$. But a relation β is transitive. So we have $C\beta F$ and hence $\mathcal{N}$ is $\mathcal{G}SCS$-module. □

Proposition 2.3. *$\mathcal{N}$ is $\mathcal{G}SCS$-module if and only if $\forall W \leq \mathcal{N}$ $\exists$ $D \leq \mathcal{N}$ and a $V \leq_s \mathcal{N} \ni D \leq_e W$ and $D \leq_e V$.*

Proof. Let $W \leq \mathcal{N}$ and $\mathcal{N}$ is $\mathcal{G}SCS$-module. $\exists$ a $V \leq_s \mathcal{N} \ni W\ \beta\ V$, $(W \cap V \leq_e W and W \cap V \leq_e V)$. Take $W \cap V = D$, so $D \leq_e W$ and $D \leq_e V$. By contrast, let $W \leq \mathcal{N}$. Then by hypothesis we have $D \leq \mathcal{N}$ and $V \leq_s \mathcal{N}$ $\ni D \leq_e W$ and $D \leq_e V$. Since $D \leq W \cap V \leq W$, $W\cap V \leq_e W$. Moreover, $D \leq W\cap V \leq V$. Then $W\cap V \leq_e V$. Hence $\mathcal{N}$ is $\mathcal{G}SCS$-module. □

Remark and Example 2.4.

1. *Any SCS-module is $\mathcal{G}SCS$-module. The cononverse does not necessarily hold: for example, letting $\mathcal{N} = Q \oplus Z_2$ as a Z-module is $\mathcal{G}SCS$-module (since $\mathcal{N}$ is $\mathcal{G}CS$-module by [8] and by next remark), it is not an SCS-module by [7].*

2. *Any $\mathcal{G}CS$-module is $\mathcal{G}SCS$-module, while any $\mathcal{G}SCS$-module is $\mathcal{G}WSCS$-module.*

3. *Any $\mathcal{G}SRCS$-module is $\mathcal{G}SCS$-module. The converse is not necessarily true: for example, letting $\mathcal{N} = Z_{12}$ as a Z-module is $\mathcal{G}SCS$-module, it is not a $\mathcal{G}SRCS$-module.*

4. *Any uniform is a $\mathcal{G}SCS$-module. The converse does not necessarily hold: for example, let $\mathcal{N} = Z_{10}$ as a Z-module is $\mathcal{G}SCS$-module, but not uniform.*

The following diagram shows the relationship of $\mathcal{G}SCS$-module to other concepts:

$$\begin{array}{c} \mathcal{G}CS - module \\ \Downarrow \\ \mathcal{G}SRCS - module \Rightarrow \mathcal{G}SCS - module \Leftarrow SCS - module \\ \Downarrow \\ \mathcal{G}WSCS - module \end{array}$$

Now in the next results we will present the conditions to make the the above observations equivalent:

Proposition 2.5. *Let $Rad(\mathcal{N}) = 0$. If $\mathcal{N}$ is $\mathcal{G}SCS$-module, then $\mathcal{N}$ is a $\mathcal{G}CS$-module.*

Proof. Consider $\mathcal{N}$ as a $\mathcal{G}SCS$-module and $F \leq_C \mathcal{N}$. Then $\exists$ a $D \leq_S \mathcal{N}$ $\ni F\beta D$. But $\mathrm{Rad}(\mathcal{N}) = 0$. Hence $D \leq_{\oplus} \mathcal{N}$. Consequently, $\mathcal{N}$ is $\mathcal{G}CS$-module. □

Proposition 2.6. *Let $\mathcal{N}$ be an R-module with $Rad\,(\mathcal{N}) = 0$. The following are equivalent:*

1. *$\mathcal{N}$ is $\mathcal{G}CS$-module.*
2. *$\mathcal{N}$ is $\mathcal{G}SCS$-module.*
3. *$\mathcal{N}$ is $\mathcal{G}WSCS$-module.*

Proof. (1) $\rightarrow$ (2) $\rightarrow$ (3) are obvious by Remarks 2.4, (3) $\rightarrow$ (2) is obvious using [9]. □

Proposition 2.7. *If $\mathcal{N}$ is $\mathcal{G}SCS$-module with $Z\,(\mathcal{N}) = 0$, then $\mathcal{N}$ is SCS-module.*

Proof. is obvious using [8]. □

From [8], if every submodule of $\mathcal{N}$ has a unique closure, then $\mathcal{N}$ is called UC-module. A submodule U of $\mathcal{N}$ is closure of $U \leq \mathcal{N}$, if $U \leq_e U \leq_c \mathcal{N}$.

Proposition 2.8. *Let $\mathcal{N}$ be a UC-module with $Rad\,(\mathcal{N}) = 0$. Then $\mathcal{N}$ is CS if and only if $\mathcal{N}$ is $\mathcal{G}SCS$-module.*

Proof. Let $\mathcal{N}$ be a CS-module. Then by [7] and Remarks 2.4 $\mathcal{N}$ is $\mathcal{G}SCS$-module. By contrast, let $\mathcal{N}$ be a $\mathcal{G}SCS$-module and $D \leq \mathcal{N}$. t Then, by definition of $\mathcal{G}SCS$-module, $\exists$ a $T \leq_s \mathcal{N} \ni D\beta T$. But $\mathcal{N}$ is UC-module. So by [8] $\alpha = \beta$. Since $\mathrm{Rad}(\mathcal{N}) = 0$, T is a summand of $\mathcal{N}$. Then $D\alpha T$ and by [8] $\mathcal{N}$ is CS-module. □

Proposition 2.9. *If $\mathcal{N}$ is $\mathcal{G}SCS$-module and $Z(\mathcal{N}) = 0$, then $\mathcal{N}$ is $\mathcal{G}SRCS$-module.*

Proof. is obvious using [1]. □

Recall that a module $\mathcal{N}$ is called supplement simple if $\mathcal{N}$ and (0) are the only supplement submodules in $\mathcal{N}$ [7].

Proposition 2.10. *If $\mathcal{N}$ is a $\mathcal{G}SCS$-module and supplement simple, then $\mathcal{N}$ is uniform.*

Proof. Let $\mathcal{N}$ be a $\mathcal{G}SCS$-module and $S \leq \mathcal{N}$. Then there is a $L \leq_s \mathcal{N} \ni S\beta L$, but $\mathcal{N}$ is supplement simple. So either $L = \mathcal{N}$ or $L = 0$, We have $S = S \cap \mathcal{N} \leq_e \mathcal{N}$. Consequently, $\mathcal{N}$ is uniform. □

Theorem 2.11. *Let $\mathcal{N}$ be supplement simple, The following are equivalent:*

1. *$\mathcal{N}$ is a $\mathcal{G}CS$-module.*
2. *$\mathcal{N}$ is a $\mathcal{G}SCS$-module.*
3. *$\mathcal{N}$ is an uniform.*

Proof. $(a) \to (b)$ is obvious by Remarks 2.4
$(b) \to (c)$ is obvious by Proposition 2.10
$(c) \to (a)$ is obvious by [8]. □

Proposition 2.12. *If $\forall W \leq_S \mathcal{N}$ and $H \leq_S \mathcal{N}$, $W \cap H \leq_S \mathcal{N}$, then every supplement (and hence summand) of $\mathcal{G}CS$-module is $\mathcal{G}CS$-module.*

Proof. This follows using the same technique of [[10], Proposition 3.8 Corollary 3.9]. □

3 Direct sum of Goldie supplement extending modules

Example 3.1. *Let $\mathcal{N} = \mathbb{Z}[x] \oplus \mathbb{Z}[x]$. As $\mathbb{Z}[x]$ is not $\mathcal{G}SCS$-module and $Rad(\mathcal{N}) = 0$ since it is not $\mathcal{G}$-extending by [8] and Proposition 2.5. But $\mathbb{Z}[x]$ is a $\mathcal{G}SCS$-module.*

Following [1], if a sequence of submodules $\{X_\rho\}$ is an independent family of $\mathcal{N}$, and $X_\rho \leq_e Y_\rho$ for any ρ, then a sequence of submodules $\{Y_\rho\}$ is an independent family and $\oplus X_\rho \leq_e \oplus Y_\rho$. $\mathcal{N}$ is called E-direct sum. Next, we need the following lemma.

Lemma 3.2. *Let $\{\mathcal{N}_\rho : \rho \in \Lambda\}$ be a family of E-direct sum modules and letX_ρ, Y_ρ be submodules of $\mathcal{N}_\rho$, for all $\rho \in \Lambda$. Then $(\oplus X_\rho)\,\beta(\oplus Y_\rho)$ if and only if $X_\rho \beta Y_\rho$ of $\mathcal{N}_\rho$ for any $\rho \in \Lambda$ [11].*

Following [12], A module N is called duo, if $g(T) \leq T, \forall T \leq N$ and $g \in$ End $(\mathcal{N})$. Now, we are ready to present the next result.

Proposition 3.3. *Let W_1 and W_2 be modules $\ni \mathcal{N} = W_1 \oplus W_2$ is a duo E-direct sum module. Then $\mathcal{N}$ is $\mathcal{G}SCS$-module if and only if W_1, W_2 are $\mathcal{G}SCS$-modules.*

Proof. Let $\mathcal{N}$ be a $\mathcal{G}SCS$-module. Then by [Proposition 2.12] W_1, W_2 are $\mathcal{G}SCS$-modules. By contrast, assume that W_1 and W_2 are $\mathcal{G}SCS$-modules and $P \leq \mathcal{N}$. But $\mathcal{N} = W_1 \oplus W_2$ is aduo module. Then by [12] $P = (P \cap W_1) \oplus (P \cap W_2)$. Since W_i is $\mathcal{G}SCS$-module for $(i = 1, 2)$ and $P \cap W_i \leq W_i$, there is a $V_i \leq_s W_i \ni (P \cap W_i)\ \beta\ V_i$. Then by Lemma 3.2 $P = (P \cap W_1) \oplus (P \cap W_2)\,\beta(V_1 \oplus V_2)$. Since $\mathcal{N}$ is E-direct sum, by [1] $V_1 \oplus V_2 \leq_s \mathcal{N}$. Hence $\mathcal{N}$ is a $\mathcal{G}SCS$-module. □

If for any $B, C \leq \mathcal{N}$ we have $F \cap (B + C) = (F \cap B) + (F \cap C)$, and if all submodules are distributive, then $\mathcal{N}$ is called a distributive module [13].

Proposition 3.4. *Let H_1, H_2 be modules $\ni \mathcal{N} = H_1 \oplus H_2$ be a distributive E-direct sum module. Then $\mathcal{N}$ is $\mathcal{G}SCS$ if and only if H_1, H_2 are $\mathcal{G}SCS$-module.*

Proof. This is similar to the previous result. □

Proposition 3.5. *If $\mathcal{N} = H_1 \oplus H_2$ be a E-direct sum R-module and ann $(H_1)+$ ann $(H_2) = R$. If H_1 and H_2 are $\mathcal{G}SCS$-module, then $\mathcal{N}$ is $\mathcal{G}SCS$-module.*

Proof. By [14] and the same technique of [[10], Proposition 4.6] the proof is complete. □

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