International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 791–796

Finslerian Projective Metrics with Small Quadratic Spheres

Ahmed Mohsin Mahdi

Department of Mathematics
Faculty of Computer Science and Information Technology
Al-Qadisiyah University
Al-Qadisiyah, Iraq

email: ahmed.m.mahdi@qu.edu.iq

(Received October 20, 2023, Accepted January 26, 2024,
Published February 12, 2024)

Abstract

If the small spheres of a Finslerian projective metric are quadrics, then it is a Riemannian projective metric of constant curvature.

1 Introduction

A metric d on an open convex non-empty domain $\mathcal{D} \subset \mathbb{R}^n$ is called *projective*, if every segment in $\mathcal{D}$ is a geodesic of d, $d(P,Q) + d(Q,R) = d(P,R)$ if and only if $Q \in \overline{PR}$, and d is continuous with respect to the Euclidean topology. Minkowski and Hilbert metrics are the most known projective metrics [5], but the set of the projective metrics is huge [12, 2, 1].

Busemann's theorem [5, 25.4] says that a Minkowskian metric on the plane is Euclidean if the circles are quadrics. In this article, we generalize this statement.

Theorem 1.1. *A Finslerian projective metric is a Riemannian projective metric of constant curvature if and only if every small sphere is a quadric.*

Key words and phrases: Projective metric, Finslerian metric, Riemannian metric, spheres.
AMS (MOS) Subject Classifications: 53C23, 53C70, 51M09, 51M10.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

2 Preliminaries

Points of $\mathbb{R}^n$ $(n \in \mathbb{N})$ are denoted as $A, B, \ldots,$ vectors are $\overrightarrow{AB}$ or $\mathbf{a}, \mathbf{b}, \ldots;$ however, we use these latter notations also for points if the origin is fixed. The open segment with endpoints A and B is denoted by $\overline{AB}$. The open ray starts from A passes through B is $\overline{A}B$, and the line through A and B is denoted by AB.

The *affine ratio* $(A, B; C)$ of the collinear points A, $B \neq A$ and $C \neq B$ satisfies $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$ [5, page 243].

Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open convex non-empty domain. Let's identify every tangent space $T_P\mathcal{D}$ with $\mathbb{R}^n$. If a projective metric d_F is such that $d_F \colon \mathcal{D} \times \mathcal{D} \ni (P, Q) \mapsto \int_0^1 F_{\mathcal{M}}(P+t(Q-P), \overrightarrow{PQ})dt \in \mathbb{R}_{\geq 0}$ holds for a *Finsler function* $F \colon \mathcal{D} \times \mathbb{R}^n \to \mathbb{R}$, then d_F is called a *Finslerian* projective metric. A point P of $\mathcal{D}$ is called *Riemannian* if the Finsler norm $F_P \colon \mathbb{R}^n \ni \mathbf{v} \mapsto F(P, \mathbf{v}) \in \mathbb{R}$ is quadratic [11]. If every point of $\mathcal{D}$ is Riemannian, then d_F is called *Riemannian* projective metric.

If a projective metric d is given, then $\mathcal{S}^{\varrho}_{d;O} = \{P : d(O, P) = \varrho\}$ is the *sphere* of radius $\varrho > 0$ with center O.

We need the following statement from [4, (16.12), p. 91]: For $n \geq 3$, if $k \in \{2, \ldots, n-1\}$, then

the border $\partial\mathcal{K}$ of a convex body $\mathcal{K} \subset \mathbb{R}^n$ is an ellipsoid if and only if every k-plane through an inner point of $\mathcal{K}$ intersects $\partial\mathcal{K}$ in a k-dimensional ellipsoid. (2.1)

3 Projective metrics of small spheres that are quadrics

The following lemma is proved here for the sake of completeness.

Lemma 3.1 Every small sphere of a Riemannian projective metric of constant curvature is a quadric.

Proof. Let $\mathcal{D} \subseteq \{(x_1, \ldots, x_n, 1) : (x_1, \ldots, x_n) \in \mathbb{R}^n\}$ be an open convex non-empty domain and let d_F be a Riemannian projective metric of constant curvature κ on $\mathcal{D}$.

If d_G is also a Riemannian projective metric of constant curvature κ on $\mathcal{D}$, then every point $P \in \mathcal{D}$ has a neighborhood $\mathcal{U} \ni P$ in $\mathcal{D}$ such that there exists an isometry $\varphi \colon \mathcal{U} \to \varphi(\mathcal{U}) \subseteq \mathcal{D}$ which satisfies $d_F(Q, R) = d_G(\varphi(Q), \varphi(R))$ for every $Q, R \in \mathcal{U}$ [8, 2.2 Corollary]. Since every such isometric mapping is a restriction of a projectivity [10, (2.1)], and every projectivity maps quadrics to quadrics, it is enough to show that the small spheres are quadrics for a well-chosen Riemannian projective metric d_G of curvature κ.

Without loss of generality, assume that $P = (0, \ldots, 0)$.

If $\kappa = 0$, let $c = 0$ and if $\kappa \neq 0$, let $c|\kappa| = \kappa$.

Equip the hypersurface $\mathcal{K}_c^n \subset \mathbb{R}^{n+1}$ of points $\mathbf{p} = (p_1, \ldots, p_n, p_{n+1})$ to satisfy $c(p_1^2 + \cdots + p_n^2) + p_{n+1}^2 = 1$ with the Riemannian metric

$$g_{c;\mathbf{p}} \colon T_{\mathbf{p}}\mathcal{K}_c^n \times T_{\mathbf{p}}\mathcal{K}_c^n \ni (\mathbf{x}, \mathbf{y}) \mapsto x_1 y_1 + \cdots + x_n y_n + c\, x_{n+1} y_{n+1}$$

at every point $\mathbf{p} \in \mathcal{K}_c^n$. Then one gets the so-called *projective model* $\bar{\mathcal{K}}_c^n$ of the space of constant curvature c [7]. The gnomonic projection of $\bar{\mathcal{K}}_c^n$ from the origin $(0, \ldots, 0)$ into the hyperplane $x_{n+1} = 1$ gives a Riemannian projective metric d_c of constant curvature c [9].

Let d_G be d_0 if $\kappa = 0$, and let d_G be $d_c/|\kappa|$ if $\kappa \neq 0$. Then d_G has constant curvature κ in $\mathcal{D}$ and the sphere $\mathcal{S}_{d_G;P}^{\varrho}$ is a Euclidean sphere in $\mathcal{D}$. Thus, the lemma follows.

Proof of Theorem 1.1. Lemma 3.1 proves the "*only if*" part of the statement. For the "*if*" part, we first prove that every point is Riemannian; i.e., that the unit sphere $\{\mathbf{v} \in T_P\mathcal{D} : F(O, \mathbf{v}) = 1\}$ in the tangent space $T_O\mathcal{D}$ is a quadric for every point $O \in \mathcal{D}$. According to (2.1), this needs to be done only in dimension two. So from now on we assume $\mathcal{D} \subseteq \mathbb{R}^2$.

Given a Finslerian projective metric d_F on a connected open bounded domain $\mathcal{D}$ of $\mathbb{R}^2$ with the Finsler function $F \colon \mathcal{D} \times \mathbb{R}^2 \to \mathbb{R}$, fix a point O and a straight line l through O, and assume that every *circle* $\mathcal{S}_{d_F;O}^{\rho}$ of small radius $\rho > 0$ is a quadric. Then $\mathcal{S}_{d_F;O}^{\rho}$ is an ellipse $\mathcal{E}_\rho$, because $\mathcal{S}_{d_F;O}^{\rho}$ is bounded.

Let C_ρ be the center of $\mathcal{E}_\rho$ and let O_ρ be the point symmetric to O in C_ρ. Let l_ρ be the straight line through C_ρ that is parallel to l, and let X_ρ be an intersection point of l_ρ and $\mathcal{E}_\rho$.

For any straight line ℓ through O, let A_ρ^ℓ and let B_ρ^ℓ be the points, where ℓ intersects $\mathcal{S}_{d_F;O}^{\rho}$. Define $A_\rho = A_\rho^{OC_\rho}$ and $B_\rho = B_\rho^{OC_\rho}$ so that $O \in \overline{C_\rho B_\rho}$. See Figure 1.

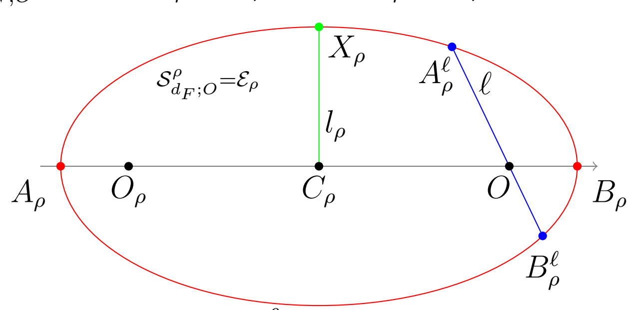

Figure 1: $\mathcal{S}_{d_F;O}^{\rho}$ is an ellipse $\mathcal{E}_\rho$

Let $\varepsilon_\rho = 1-(O,C_\rho;B_\rho)$, $a_\rho = \rho/(1-\varepsilon_\rho^2)$, and $c_\rho = a_\rho\varepsilon_\rho$.

Let d_ρ be the Euclidean metric satisfying $d_\rho(C_\rho,B_\rho)=a_\rho$ (hence $d_\rho(C_\rho,O) = c_\rho$), and $d_\rho^2(C_\rho,X_\rho)=a_\rho^2-c_\rho^2$. Then we get $\mathcal{E}_\rho = \{E\in\mathbb{R}^2 : 2a_\rho = d_\rho(O,E)+d_\rho(E,O_\rho)\}$.

So, for any straight line ℓ through O, we have

$$\begin{aligned}\frac{d_F(A_\rho^\ell,O)}{d_\rho(A_\rho^\ell,O)}+\frac{d_F(B_\rho^\ell,O)}{d_\rho(B_\rho^\ell,O)} &= \frac{\rho}{d_\rho(A_\rho^\ell,O)}+\frac{\rho}{d_\rho(B_\rho^\ell,O)}\\ &= \rho\Big(\frac{1}{a_\rho-c_\rho}+\frac{1}{a_\rho+c_\rho}\Big) = \frac{2a_\rho\rho}{a_\rho^2-c_\rho^2} = \frac{2\rho/a_\rho}{1-\varepsilon_\rho^2} = 2,\end{aligned} \tag{3.1}$$

where the second equation follows from the polar form of the ellipse $\mathcal{E}_\rho$ relative to focus O.

Fix a $\varrho>0$. By the Busemann–Mayer theorem [6, Theorem 4.3] we have

$$\begin{aligned}\lim_{\rho\to0}\frac{d_F(A_\rho^\ell,O)}{d_\rho(A_\rho^\ell,O)} &= \lim_{\rho\to0}\Big(\frac{d_F(A_\rho^\ell,O)}{d_\varrho(A_\rho^\ell,O)}\frac{d_\varrho(A_\rho^\ell,O)}{d_\rho(A_\rho^\ell,O)}\Big)\\ &= F\Big(O,\lim_{\rho\to0}\frac{A_\rho^\ell-O}{d_\varrho(A_\rho^\ell,O)}\Big)\lim_{\rho\to0}\frac{d_\varrho(A_\rho^\ell,O)}{d_\rho(A_\rho^\ell,O)} = F\Big(O,\lim_{\rho\to0}\frac{A_\rho^\ell-O}{d_\rho(A_\rho^\ell,O)}\Big),\end{aligned}$$

and in a similar way we also have

$$\lim_{\rho\to0}\frac{d_F(B_\rho^\ell,O)}{d_\rho(B_\rho^\ell,O)} = F\Big(O,\lim_{\rho\to0}\frac{O-B_\rho^\ell}{d_\rho(B_\rho^\ell,O)}\Big).$$

Since $\frac{A_\rho^\ell-O}{d_\rho(A_\rho^\ell,O)} = \frac{O-B_\rho^\ell}{d_\rho(B_\rho^\ell,O)}$, equation (3.1) gives

$$1 = F\Big(O,\lim_{\rho\to0}\frac{A_\rho^\ell-O}{d_\rho(A_\rho^\ell,O)}\Big).$$

Thus, the unit circle of the Finsler norm $F_O(\cdot) = F(O,\cdot)$ is the limit of the unit circles of d_ρ. This means that the closed quadrics $\mathcal{S}^1_{d_\rho;O}$ converge to the strictly convex closed curve, the unit circle of F_O. Hence the unit circle of F_O is a quadric; i.e., an ellipse.

Thus, d_F is a Riemannian projective metric-space.

By Beltrami's theorem [3] (see also [5, (29.3)]), every Riemannian projective metric has constant curvature, so the proof is complete.

Theorem 1.1 can be sharpened for special projective metrics. It is well-known that a Minkowski geometry is a model of the Euclidean geometry if and only if it has *one* Riemannian point [5, 24.10 with 25.4], and it turned out recently [11] that a Hilbert geometry with twice differentiable boundary in the plane is a Cayley–Klein model of the hyperbolic geometry if and only if it has *two* Riemannian points.

Acknowledgment. The author appreciates Árpád Kurusa's assistance in preparing and discussing all parts of this article.

References

[1] Ralph Alexander, Planes for which the lines are the shortest paths between points, Illinois Journal of Mathematics, **22,** no. 2, (1978), 177–190.

[2] Rouben V. Ambartzumian, A note on pseudo-metrics on the plane, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, **37,** no. 2, (1976), 145–155.

[3] Eugenio Beltrami, Risoluzione del problema riportare i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette, 1865.

[4] Herbert Busemann, The geometry of geodesics, Academic Press Inc., New York, 1955.

[5] Herbert Busemann, Paul J. Kelly, Projective Geometry and Projective Metrics, Academic Press Inc., New York, 1953.

[6] Herbert Busemann, Walther Mayer, On the foundations of calculus of variations, Transactions of the American Mathematical Society, **49,** no. 2, (1941), 173–198.

[7] James W. Cannon, William J. Floyd, Richard Kenyon, Walter R. Parry, Hyperbolic geometry, Flavors of geometry, MSRI Publications, **31,** (1997), 59–115.

[8] Manfredo P. do Carmo, Riemannian Geometry, 1992.

[9] Árpád Kurusa, Support theorems for totally geodesic Radon transforms on constant curvature spaces, Proceedings of the American Mathematical Society, **112,** no. 2, (1994), 429–435.

[10] Árpád Kurusa, Straight projective-metric spaces with centres, Journal of Geometry, **109,** no. 1, (2018), 22.

[11] Árpád Kurusa, Hilbert geometries with Riemannian points, Annali di Matematica Pura ed Applicata, **199,** no. 2, (2020), 809–820.

[12] A. V. Pogorelov, A complete solution of Hilberts fourth problem, **14,** (1973), 46–49.