International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 573–584

On the distance spectrum of cozero-divisor graph

Magi P M

Department of Mathematics
Panampilly Memorial Govt. College, Chalakkudy
University of Calicut
Kerala, India

email: tcr.maggie81@gmail.com

(Received October 24, 2023, Accepted November 24, 2023,
Published February 12, 2024)

Abstract

For a commutative ring R with unity, the cozero-divisor graph denoted by $\Gamma'(R)$, is an undirected simple graph whose vertex set is the set of all non-zero and non-unit elements of R. Two distinct vertices x and y are adjacent if and only if x does not belong to the ideal Ry and y does not belong to Rx. The cozero-divisor graph on the ring of integers modulo n is a generalized join of its induced sub graphs all of which are null graphs. This property of the cozero-divisor graph on $\mathbb{Z}_n$ is used in finding its distance spectrum. In this paper, the distance matrix of the cozero-divisor graph on the ring of integers modulo n is discovered and the general method is discussed to find its distance spectrum, for any value of n. Also, the distance spectrum of this graph is explored for some values of n, by means of the vertex weighted distance matrix of the co-proper divisor graph of n.

1 Introduction

Spectral Graph Theory is an emerging and flourishing area in Graph Theory which studies the relation between graph properties and the spectrum of

Key words and phrases: Distance matrix, cozero-divisor graph, distance spectrum.
AMS (MOS) Subject Classifications: 05C50.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

graph theoretic matrices like adjacency matrix, Laplacian matrix, distance matrix, etc. It is a recent trend that graphs are crafted out of algebraic structures like groups and rings. In-depth research has been carried out in classifying the rings on the structural properties of these derived algebraic graphs. For example, a graph can be associated to a commutative ring R with unity, by considering its non-zero zero-divisors as its vertices and connecting two of them by an edge if their product is zero. This graph is called the zero-divisor graph of R, denoted by $\Gamma(R)$. See [1, 11, 13, 14, 16] and the references therein for the vast literature on the study of zero-divisor graphs. Afkhami et al. [5] introduced the cozero-divisor graph of a commutative ring, in which they have studied the basic graph-theoretic properties including completeness, girth, clique number, etc. of the cozero-divisor graph. The cozero-divisor graph of a ring R with unity, denoted by $\Gamma'(R)$, is an undirected simple graph whose vertex set is the set of all non-zero and non-unit elements of R and two distinct vertices x and y are adjacent if and only if $x \notin Ry$ and $y \notin Rx$. For a finite commutative ring R, every non-zero element is either a unit or a zero divisor. Thus, for any finite commutative ring, the vertex set of its co-zero divisor graph consists of all non-zero zero-divisors. The complement of the cozero-divisor graph and the characterization of the commutative rings with forest, star, or unicyclic cozero-divisor graphs have been investigated in [3].

2 Basic definitions and notations

A graph G is an ordered triple $G = (V(G), E(G), \psi(G))$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$ of edges and an incidence function $\psi(G)$ which associates with each element of $E(G)$, an unordered pair of vertices (not necessarily distinct) of G. A graph which has no loops and multiple edges is called a simple graph. A graph is trivial if it has only one point. A graph G is complete if every pair of distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n. An empty graph (null graph) is a graph which has no edges. A graph G is said to be complete bipartite if G is simple, bipartite with bipartition (X, Y) and each vertex of X is joined to every vertex of Y. If $\mid X \mid= m, \mid Y \mid= n$, then G is denoted by $K_{m,n}$. Let V^1 be a non-empty subset of the vertex set V of G. The subgraph of G whose vertex set is V^1 and whose edge set is the set of those edges of G that have both ends in V^1 is called the subgraph of G induced by V^1 and is denoted by $G[V^1]$. We say $G[V^1]$ is the induced subgraph of G. If $u \in V(G)$, the open neighborhood of u; denoted by $N_G(u)$ is the set of vertices adjacent to u in G. We denote by $\delta(G)$ and $\Delta(G)$, the minimum degree and the maximum degree of vertices in G, respectively. A graph G is k-regular if $deg(v) = k$ for all $v \in V$. A regular graph is a graph which is k-regular for some $k \geq 0$. If u and v are distinct vertices in a graph G, $d_G(u, v)$ denotes the distance between u and v; which is the length of a shortest path between u and v. Clearly, $d_G(u, u) = 0$ and $d_G(u, v) = \infty$ if there is no path between u and v. The distance matrix of a simple connected graph G of order n is the symmetric matrix $D = (d_{i,j})_{n\times n}$; the rows and columns are labeled by vertices, where $d_{i,j=}d_G(u_i, u_j)$, $i \neq j$ and $d_{i,j} = 0$ if $i = j$.

Let G be a finite graph with vertices labeled as $1, 2, 3, ..., n$ and let $H_1, H_2, ..., H_n$ be a family of vertex disjoint graphs. The G join of $H_1, H_2, ..., H_n$ denoted by $\bigvee_G\{H_i : 1 \leq i \leq n\}$ is obtained by replacing each vertex i of G by the graph H_i and inserting all or none of the possible edges between H_i and H_j depending on whether or not i and j are adjacent in G.

The basic definitions in graph theory are standard and are from [10]. Refer [7] for results in Spectral Graph Theory. An eigenvalue of a matrix is simple, if its algebraic multiplicity is 1. For a real symmetric matrix, all eigenvalues are real and the algebraic multiplicity of each eigenvalue is same as its geometric multiplicity. A graph is said to be integral if all the eigenvalues are integers. For a natural number n, $\phi(n)$ is the number of positive integer less than n and relatively prime to n. In this paper, J denotes an all-one matrix and O denotes a zero matrix. $\mathbf{1}_n$ denotes the all-one column vector of order $n \times 1$, and I_n denotes the unit matrix of order n.

3 Structure of the cozero-divisor graph $\Gamma'(\mathbb{Z}_n)$

By a proper divisor of n, we mean a positive divisor d such that d/n , $1 < d < n$. Let $\xi(n)$ denote the number of proper divisors of n. Then, $\xi(n) = \sigma_0(n)-2$, where $\sigma_k(n)$ is the sum of k powers of all divisors of n, including n and 1. It is convenient to denote the proper divisors of n by $d_1, d_2, ..., d_{\xi(n)}$. Consider the canonical decomposition $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$, where $p_1, p_2, ..., p_r$ are distinct primes, and $r, n_1, n_2, ..., n_r$ are positive integers. Then,

$$\xi(n) = \prod_{i=1}^{r}(n_i + 1) \ \ - 2.$$

Let $\mathcal{A}(d) = \{k \in \mathbb{Z}_n : gcd(k,n) = d\}$. Then $\{\mathcal{A}(d_1), \mathcal{A}(d_2), ..., \mathcal{A}(d_{\xi(n)})\}$ is an equitable partition for the vertex set of $\Gamma'(\mathbb{Z}_n)$ such that $\mathcal{A}(d_i) \cap \mathcal{A}(d_j) = \phi, i \neq j$

Lemma 3.1. *[13]* $|\mathcal{A}(d_i)| = \phi(\frac{n}{d_i})$, *for every* $i = 1, 2, ...\xi(n)$.

Remark 3.2. *For* $x; y \in \mathcal{A}(d_i)$, *we have* $\langle x\rangle = \langle y\rangle = \langle d_i\rangle$.

Lemma 3.3. *[4] Let* $x \in \mathcal{A}(d_i), y \in \mathcal{A}(d_j), i \neq j$, *where* $i; j \in \{1, 2, ..., \xi(n)\}$. *Then* x *is adjacent to* y *in* $\Gamma'(\mathbb{Z}_n)$ *if and only if* d_i/d_j *and* d_j *does not divide* d_i.

Remark 3.4. *For* $x; y \in \mathcal{A}(d_i)$, *we have* $\langle x\rangle = \langle y\rangle = \langle d_i\rangle$. *Thus, for distinct vertices* $x; y$ *of* $\mathcal{A}(d_i)$, $x \in \langle y\rangle$ *and* $y \in \langle x\rangle$, *it follows that any two distinct* x *and* y *in* $\mathcal{A}(d_i)$ *are non adjacent in* $\Gamma'(\mathbb{Z}n)$ *and so the subgraph of* $\Gamma'(\mathbb{Z}n)$ *induced by* $\mathcal{A}(d_i)$ *is a null graph with* $\phi(\frac{n}{d_i})$ *number of vertices for every* $i = 1, 2, ...\xi(n)$.

The definition of the proper divisor graph of n which is closely associated with the zero divisor graph $\Gamma(\mathbb{Z}_n)$ in describing its joined union structure, is given below.

Definition 3.5. *[9] The proper divisor graph of* n, *denoted by* Υ_n *is a simple connected graph with vertices labeled as* $d_1, d_2, ..., d_{\xi(n)}$, *in which two distinct vertices* d_i *and* d_j *are adjacent if and only if* $n/d_i d_j$.

Analogously, the co-proper divisor graph denoted by by $\Upsilon'(n)$ is defined as follows.

Definition 3.6. *The co-proper divisor graph* $\Upsilon'(n)$ *is the simple undirected graph whose vertices are labeled as the proper divisors* $d_1, d_2, ..., d_{\xi(n)}$ *of* n *and any two distinct vertices* d_i *and* d_j *are adjacent if and only if* d_i *does not divide* d_j *and* d_j *does not divide* d_i.

$\Upsilon'(n)$ is connected if and only if $n \neq p^k$ for any prime p and $k \geq 3$.

Lemma 3.7. *[4]* $\Gamma'(\mathbb{Z}_n) = \Upsilon'_n \left[\Gamma'(\mathcal{A}(d_1)), \Gamma'(\mathcal{A}(d_2)), ..., \Gamma'(\mathcal{A}(d_{\xi(n)}))\right]$

For example, consider $\Gamma'(\mathbb{Z}_{36})$. The number of proper divisors of 36 is 7. They are precisely $2, 3, 4, 6, 9, 12, 18$. The non-zero divisors of $\Gamma'(\mathbb{Z}_{36})$ are partitioned into 7 classes as follows. $\mathcal{A}(2) = \{2, 10, 14, 22, 26, 34\}$, $\mathcal{A}(3) = \{3, 15, 21, 33\}$, $\mathcal{A}(4) = \{4, 8, 16, 20, 28, 32\}$, $\mathcal{A}(6) = \{6, 30\}$,

$\mathcal{A}(9) = \{9, 27\}, \quad \mathcal{A}(12) = \{12, 24\}, \quad \mathcal{A}(18) = \{18\}$.
The graphs $\Gamma(\mathbb{Z}_{36})$ and Υ'_{36} are given in Figure 4.2 and Figure 4.3. Note that the subgraphs induced by these partition classes, that is $\Gamma'(\mathcal{A}(2)), \Gamma'(\mathcal{A}(3)), \Gamma'(\mathcal{A}(4)), \Gamma'(\mathcal{A}(6)), \Gamma'(\mathcal{A}(9)), \Gamma'(\mathcal{A}(12))$ and $\Gamma'(\mathcal{A}(18)))$ are all null graphs.

4 Spectrum of the generalized join of regular graphs

For $i \in \{1, 2, ..., k\}$, let M_i be an $n_i \times n_i$ symmetric matrix, such that the all one vector $\mathbf{1}_{n_i}$ is an eigenvector for the eigenvalue μ_i of G_i. That is $M_i \mathbf{1}_{n_i} = \mu_i \mathbf{1}_{n_i}$. For an arbitrary $\frac{k(k-1)}{2}$ number of reals, $\rho_{i,j}, 1 \leq i < j \leq k$ consider the symmetric matrix

$$M = \begin{bmatrix} M_1 & \rho_{1,2}\mathbf{1}_{n_1}\mathbf{1}_{n_2}^T & \rho_{1,3}\mathbf{1}_{n_1}\mathbf{1}_{n_3}^T & \cdots & \rho_{1,k}\mathbf{1}_{n_1}\mathbf{1}_{n_k}^T \\ \rho_{1,2}\mathbf{1}_{n_2}\mathbf{1}_{n_1}^T & M_2 & \rho_{2,3}\mathbf{1}_{n_2}\mathbf{1}_{n_3}^T & \cdots & \rho_{2,k}\mathbf{1}_{n_2}\mathbf{1}_{n_k}^T \\ \rho_{1,3}\mathbf{1}_{n_3}\mathbf{1}_{n_1}^T & \rho_{2,3}\mathbf{1}_{n_3}\mathbf{1}_{n_2}^T & M_3 & \cdots & \rho_{3,k}\mathbf{1}_{n_3}\mathbf{1}_{n_k}^T \\ \vdots & \vdots & & \ddots & \vdots \\ \rho_{1,k-1}\mathbf{1}_{n_{k-1}}\mathbf{1}_{n_1}^T & \rho_{2,k-1}\mathbf{1}_{n_{k-1}}\mathbf{1}_{n_2}^T & \cdots & M_{k-1} & \rho_{k-1,k}\mathbf{1}_{n_{k-1}}\mathbf{1}_{n_k}^T \\ \rho_{1,k}\mathbf{1}_{n_k}\mathbf{1}_{n_1}^T & \rho_{2,k}\mathbf{1}_{n_k}\mathbf{1}_{n_2}^T & \cdots & \cdots & M_k \end{bmatrix} \tag{4.1}$$

and the matrix

$$F_k = \begin{bmatrix} \mu_1 & \rho_{1,2}\sqrt{n_1 n_2} & \cdots & \rho_{1,k-1}\sqrt{n_1 n_{k-1}} & \rho_{1,k}\sqrt{n_1 n_k} \\ \rho_{1,2}\sqrt{n_1 n_2} & \mu_2 & \cdots & \rho_{2,k-1}\sqrt{n_2 n_{k-1}} & \rho_{2,k}\sqrt{n_2 n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1,k-1}\sqrt{n_1 n_{k-1}} & \rho_{2,k-1}\sqrt{n_2 n_{k-1}} & \cdots & \mu_{k-1} & \rho_{k-1,k}\sqrt{n_{k-1} n_k} \\ \rho_{1,k}\sqrt{n_1 n_k} & \rho_{2,k}\sqrt{n_2 n_k} & \cdots & \rho_{k-1,k}\sqrt{n_{k-1} n_k} & \mu_k \end{bmatrix}_{k \times k} \tag{4.2}$$

Theorem 4.1. *[6]* $\sigma(M) = \bigcup_{i=1}^{k} (\sigma(M_i) \setminus \{\mu_i\}) \cup \sigma(F_k)$

5 Distance spectrum of $\Gamma'(\mathbb{Z}_n)$

The graph $\Gamma'(\mathbb{Z}_4)$ is a trivial graph and $\Gamma'(\mathbb{Z}_{p^t})$, for any prime p and any positive integer $t \geq 2$ (except $n = 4$) is a totally disconnected graph. So we avoid these values of n for computing the distance eigenvalues of $\Gamma'(\mathbb{Z}_n)$. Since $\Gamma'(\mathbb{Z}_n)$ has a generalized join structure, the distance between the vertices of the joining graph $\Upsilon'(n)$ is an important key to determine the entries of the distance matrix of $\Gamma'(\mathbb{Z}_n)$. Also, the subgraph induced by $\mathcal{A}(d_i)$, for every proper divisor d_i of n, is a null graph of order $\phi(\frac{n}{d_i})$ (denoted by $\Gamma'(\mathcal{A}(d_i))$), and thus the distance between the vertices among $\Gamma'(\mathcal{A}(d_i))$ is 2 and this is true for $i = 1, 2, ..., \xi(n)$.

Theorem 5.1. *Let $d_1, d_2, ..., d_{\xi(n)}$ be the proper divisors of n. Let $d_{i,j}$ be the distance between the i^{th} and j^{th} vertex of Υ'_n. Then, the distance matrix of the co-zero divisor graph $\Gamma'(\mathbb{Z}_n)$ is given by*
$D(\Gamma'(\mathbb{Z}_n)) =$

$$\begin{bmatrix} M_1 & d_{1,2}J_{m_1\times m_2} & d_{1,3}J_{m_1\times m_3} & \cdots & d_{1,k}J_{m_1\times m_{\xi(n)}} \\ d_{1,2}J^T_{m_1\times m_2} & M_2 & d_{2,3}J_{m_2\times m_3} & \cdots & d_{2,\xi(n)}J_{m_2\times m_{\xi(n)}} \\ d_{1,3}J^T_{m_1\times m_3} & d_{2,3}J^T_{m_2\times m_3} & M_3 & \cdots & d_{3,\xi(n)}J_{m_3\times m_{\xi(n)}} \\ \vdots & \vdots & & \ddots & \vdots \\ d_{1,\xi(n)-1}J^T_{m_1\times m_{\xi(n)-1}} & d_{2,\xi(n)-1}J^T_{m_2\times m_{\xi(n)-1}} & \cdots & M_{\xi(n)-1} & d_{\xi(n)-1,\xi(n)}J_{m_{\xi(n)-1}\times m_{\xi(n)}} \\ d_{1,\xi(n)}J^T_{m_1\times m_{\xi(n)}} & d_{2,\xi(n)}J^T_{m_2\times m_{\xi(n)}} & \cdots & \cdots & M_{\xi(n)} \end{bmatrix} \tag{5.3}$$

where $m_i = \phi(\frac{n}{d_i})$ and $M_i = 2(J-I)_{m_i}$ for $i = 1, 2, ..., \xi(n)$.

Theorem 5.2. *For any n,let $d_1, d_2, ..., d_{\xi(n)}$ be the proper divisors. Then, the co-zero divisor graph $G = \Gamma'(\mathbb{Z}_n)$ has -2 as a distance eigenvalue with multiplicity $n - \phi(n) - \xi(n) - 1$ and the remaining distance eigenvalues are the eigenvalues of the matrix given below*

$$W_D(\Upsilon'_n) = \begin{bmatrix} 2(\phi(\frac{n}{d_1})-1) & d_{1,2}\phi(\frac{n}{d_2}) & \cdots & d_{1,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ d_{1,2}\phi(\frac{n}{d_1}) & 2(\phi(\frac{n}{d_2})-1) & \cdots & d_{2,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,\xi(n)}\phi(\frac{n}{d_1}) & d_{2,\xi(n)}\phi(\frac{n}{d_2}) & \cdots & 2(\phi(\frac{n}{d_{\xi(n)}})-1) \end{bmatrix}$$

Proof. Let $G = \Gamma'(\mathbb{Z}_n)$. The subgraphs $\Gamma'(\mathcal{A}_{d_i})$ for $i = 1, 2, ..., \xi(n)$ are all null graphs and hence regular with regularity index 0. The diagonal blocks M_i in the distance matrix $D(G)$ given in equation (5.3), corresponding to $\Gamma'(\mathcal{A}_{d_i})$ is $2(J-I)$ of order $m_i = \phi(\frac{n}{d_i})$, for $i = 1, 2, ..., \xi(n)$. Also, the other blocks $d_{i,j}J_{m_i\times m_j}, 1 \leq i < j \leq \xi(n)$ can be realized as $\mathbf{1}_{m_i}\mathbf{1}^T_{m_j}$. The eigenvalues of $M_i = 2(J-I)$ are -2 and $2(\phi(\frac{n}{d_i})-1)$ with multiplicities $\phi(\frac{n}{d_i})-1$ and 1 respectively and the eigenvalue $2(\phi(\frac{n}{d_i})-1)$ is the Perron eigenvalue of M_i with eigenvector $\mathbf{1}_{\phi(\frac{n}{d_i})}$ for $i = 1, 2, ..., \xi(n)$. Thus, applying Theorem 4.1, where $\rho_{i,j} = d_{i,j}$, $1 \leq i < j \leq \xi(n)$ which is the distance between the i^{th} and j^{th} vertex of Υ'_n and $\mu_i = 2(\phi(\frac{n}{d_i})-1)$, it can be easily seen that $-2(\phi(\frac{n}{d_i})-1)$ is an eigenvalue of $D(G)$ with multiplicity $\sum_{i=1}^{\xi(n)}(\phi(\frac{n}{d_i})-1)$ and the remaining eigenvalues are those of the matrix

$$T = \begin{bmatrix} 2(\phi(\frac{n}{d_1}) - 1) & d_{1,2}\sqrt{m_1 m_2} & \dots & d_{1,\xi(n)}\sqrt{m_1 m_{\xi(n)}} \\ d_{1,2}\sqrt{m_1 m_2} & 2(\phi(\frac{n}{d_2}) - 1) & \dots & d_{2,\xi(n)}\sqrt{m_2 m_{\xi(n)}} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,\xi(n)}\sqrt{m_1 m_{\xi(n)}} & d_{2,\xi(n)}\sqrt{m_2 m_{\xi(n)}} & \dots & 2(\phi(\frac{n}{d_{\xi(n)}}) - 1) \end{bmatrix} \tag{5.4}$$

Also,

$$\begin{aligned} \sum_{i=1}^{\xi(n)}(\phi(\frac{n}{d_i}) - 1) &= \sum_{d/n,1<d<n} \Big(\phi(\frac{n}{d}) - 1\Big) \\ &= \sum_{d/n,1<d<n} \phi(\frac{n}{d}) - \xi(n) \\ &= \sum_{d/n,1\leq d\leq n} \phi(\frac{n}{d}) - \phi(n) - \phi(1) - \xi(n) \\ &= n - \phi(n) - 1 - \xi(n) \end{aligned}$$

Consider the graph Υ'_n as a vertex weighted graph by assigning the weight $m_i = \phi(\frac{n}{d_i})$ to its i^{th} vertex, for $i = 1, 2, ..., \xi(n)$ and consider the diagonal matrix of vertex weights,

$$W = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & m_{\xi(n)} \end{bmatrix}.$$

It can be easily seen that

$$W^{-\frac{1}{2}}TW^{\frac{1}{2}} = \begin{bmatrix} 2(\phi(\frac{n}{d_1}) - 1) & d_{1,2}\phi(\frac{n}{d_2}) & \dots & d_{1,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ d_{1,2}\phi(\frac{n}{d_1}) & 2(\phi(\frac{n}{d_2}) - 1) & \dots & d_{2,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,\xi(n)}\phi(\frac{n}{d_1}) & d_{2,\xi(n)}\phi(\frac{n}{d_2}) & \dots & 2(\phi(\frac{n}{d_{\xi(n)}}) - 1) \end{bmatrix} \tag{5.5}$$

Thus, the matrix T and the matrix on the right side of equation (5.5) are similar and so they have same spectrum. □

Remark 5.3. *-2 is a distance eigenvalue of $\Gamma'(\mathbb{Z}_n)$ for all values of n and the remaining distance eigenvalues are determined by the vertex weighted combinatorial distance matrix $W_D(\Upsilon'_n)$ associated to Υ'_n. Thus in order to get the distance spectrum of $\Gamma'(\mathbb{Z}_n)$ of any value of n, however large it may be, we need look into a smaller matrix $W_D(\Upsilon'_n)$ of order $\xi(n)$.*

Corollary 5.4. *The graph $\Gamma'(\mathbb{Z}_n)$ is distance integral if and only if the matrix $W_D(\Upsilon'_n)$ is integral.*

Corollary 5.5. *Let $p < q$ be two distinct primes. The distance spectrum of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{pq})$ is given by $(\sigma D(\Gamma'(\mathbb{Z}_{pq}))) =$*

$$\begin{pmatrix} -2 & p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1} & p+q-4-\sqrt{p^2+q^2-pq-(p+q)+1} \\ p+q-4 & 1 & 1 \end{pmatrix}$$

Proof. Consider $\Gamma'(\mathbb{Z}_{pq})$, where $p < q$ are distinct primes. The proper divisors of pq are $p$and q. Since p and q do not divide each other, the co-proper divisor zero graph $\Upsilon'_{pq} \equiv K_2$, with vertices labeled as p and q. Clearly, $\Gamma'(\mathbb{Z}_{pq}) = K_2[\Gamma'(\mathcal{A}(p)), \Gamma'(\mathcal{A}(q))]$, where $\Gamma'(\mathcal{A}(p)) = \overline{K}_{q-1}$ and $\Gamma'(\mathcal{A}(q)) = \overline{K}_{p-1}$. Using Theorem 5.1, the distance matrix of $\Gamma'(\mathbb{Z}_{pq})$ is given by

$$D(\Gamma'(\mathbb{Z}_{pq})) = \left[\begin{array}{c|c} 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & 2(J-I)_{(p-1)\times(p-1)} \end{array}\right]$$

Thus, using Theorem 5.2 , we see that -2 is an eigenvalue of $\Gamma(\mathbb{Z}_{pq})$ with multiplicity $p+q-4$ And, the other distance eigenvalues of $\Gamma(\mathbb{Z}_{pq})$ is determined by its vertex weighted distance matrix,

$$W_D(\Upsilon'_{pq}) = \begin{bmatrix} 2(q-2) & p-1 \\ q-1 & 2(p-2) \end{bmatrix}$$

So the remaining two distance eigenvalues of this graph are determined by the polynomial, $Q(x) = x^2 - 2x(p+q-4) + 3pq - 7(p+q) + 15$. Thus,

$$\sigma(D(\Gamma'(\mathbb{Z}_{pq}))) = \begin{pmatrix} -2 & p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1} & p+q-4-\sqrt{p^2+q^2-pq-(p+q)+1} \\ p+q-4 & 1 & 1 \end{pmatrix}$$

□

Corollary 5.6. *Let $p < q$ be two distinct primes. -2 ia s distance spectrum of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{p^2q})$ with multiplicity $p^2q - p(p-1)(q-1) - 5$ and the remaining distance eigenvalues are the eigenvalues of the matrix,*

$$W_D(\Upsilon'_{p^2q}) = \begin{bmatrix} 2(pq-p-q) & p(p-1) & 2(q-1) & 2(p-1) \\ (p-1)(q-1) & 2(p^2-p-1) & (q-1) & 2(p-1) \\ 2(p-1)(q-1) & p(p-1) & 2(q-2) & (p-1) \\ 2(p-1)(q-1) & 2p(p-1) & (q-1) & 2(p-2) \end{bmatrix}$$

Proof. Consider $\Gamma'(\mathbb{Z}_{p^2q})$, where $p < q$ are distinct primes. The proper divisors of p^2q are p, qp^2 and pq. The co-proper divisor zero graph $\Upsilon'_{p^2q} \equiv P_4$, as shown in Figure:**1**, from which the distance between any two distinct vertices are clear.

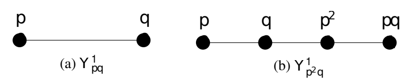

Figure 1: The co-proper divisor graphs Υ^1_{pq} and $\Upsilon^1_{p^2q}$

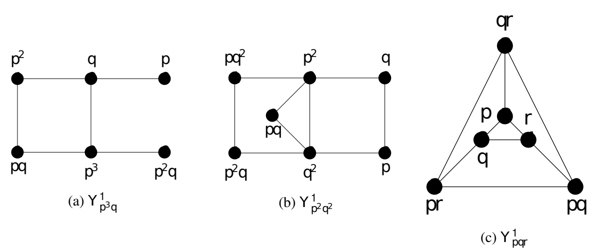

Figure 2: The co-proper divisor graphs $\Upsilon^1_{p^3q}$, $\Upsilon^1_{p^2q^2}$ and Υ^1_{pqr}

1

Also,
$\Gamma'(\mathbb{Z}_{p^2q}) = P_4[\Gamma'(\mathcal{A}(p)), \Gamma'(\mathcal{A}(q)), \Gamma'(\mathcal{A}(p^2)), \Gamma'(\mathcal{A}(pq))]$, where $\Gamma'(\mathcal{A}(p)) = \overline{K}_{(p-1)(q-1)}$, $\Gamma'(\mathcal{A}(q)) = \overline{K}_{p(p-1)}$, $\Gamma'(\mathcal{A}(p^2)) = \overline{K}_{(q-1)}$, $\Gamma'(\mathcal{A}(pq)) = \overline{K}_{(p-1)}$. Thus using Theorem 5.2 , we see that -2 is an eigenvalue of $\Gamma(\mathbb{Z}_{p^2q})$ with multiplicity $p^2q - p(p-1)(q-1) - 5$ And, the other four remaining distance eigenvalues of $\Gamma(\mathbb{Z}_{p^2q})$ are determined by the vertex weighted distance matrix, associated to Υ'_{p^2q} given by

$$W_D(\Upsilon'_{p^2q}) = \begin{bmatrix} 2(pq-p-q) & p(p-1) & 2(q-1) & 2(p-1) \\ (p-1)(q-1) & 2(p^2-p-1) & (q-1) & 2(p-1) \\ 2(p-1)(q-1) & p(p-1) & 2(q-2) & (p-1) \\ 2(p-1)(q-1) & 2p(p-1) & (q-1) & 2(p-2) \end{bmatrix}$$

□

The co-proper divisor graphs of $n = p^3q, p^2q^2, pqr$ for distinct primes p, q, r are given in Figure:**2**. As proved above, we have the following corollaries.

Corollary 5.7. *Let $p < q$ be two distinct primes. Then, -2 ia s distance spectrum of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{p^3q})$ with multiplicity $p^3q - p^2(p-1)(q-1) - 7$ and the remaining distance eigenvalues are the eigenvalues of the matrix,*

$$W_D(\Upsilon'_{p^3q}) =$$

$$\begin{bmatrix} 2\left(p(p-1)(q-1)-1\right) & p^2(p-1) & 2(p-1)(q-1) & 3p(p-1) & 2(q-1) & 3(p-1) \\ p(p-1)(q-1) & 2\left(p^2(p-1)-1\right) & (p-1)(q-1) & 2p(p-1) & q-1 & 2(p-1) \\ 2p(p-1)(q-1) & p^(p-1) & 2\left(pq-p-q\right) & p(p-1) & 2(q-1) & 3(p-1) \\ 3p(p-1)(q-1) & 2p^2(p-1) & (p-1)(q-1) & 2\left(p(p-1)-1\right) & q-1 & 2(p-1) \\ 2p(p-1)(q-1) & p^2(p-1) & 2(p-1)(q-1) & p(p-1) & 2(q-2) & p-1 \\ 3p(p-1)(q-1) & 2p^2(p-1) & 3(p-1)(q-1) & 2p(p-1) & q-1 & 2(p-2) \end{bmatrix}$$

Corollary 5.8. *Let $p < q < r$ be three distinct primes. Then, -2 ia s distance spectrum of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{pqr})$ with multiplicity $pqr - (p-1)(q-1)(r-1) - 7$ and the remaining distance eigenvalues are the eigenvalues of the matrix,*

$$W_D(\Upsilon'_{pqr}) = \begin{bmatrix} 2\left(qr-q-r\right) & m_2 & m_3 & m_4 & 2m_5 & 2m_6 \\ m_1 & 2\left(pr-p-r\right) & m_3 & 2m_4 & 2m_5 & m_6 \\ m_1 & m_2 & 2\left(pq-p-q\right) & 2m_4 & m_5 & 2m_6 \\ m_1 & 2m_2 & 2m_3 & 2(p-2) & m_5 & m_6 \\ 2m_1 & 2m_2 & m_3 & m_4 & 2(r-2) & m_6 \\ 2m_1 & m_2 & 2m_3 & m_4 & m_5 & 2(q-2) \end{bmatrix}$$

where $m_1 = (q-1)(r-1), m_2 = (p-1)(r-1), m_3 = (p-1)(q-1), m_4 = p-1$, $m_5 = r-1, m_6 = q-1$.

Conclusion

The computation of distance eigenvalues of the cozero-divisor graph on the ring of integers modulo n is made easy due to its joined union structure. The computation completely depends upon its joining graph, the co-proper divisor graph Υ'_n.

References

[1] D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra, **217,** no. 2, (1999), 434–447.

[2] M. Aouchiche, P. Hansen, Distance spectra of graphs, A survey; Linear Algebra Appl. **458,** (2014), 301–386.

[3] M. Afkhami, K. Khashyarmanesh. On the cozero-divisor graphs of commutative rings and their complements, Bull. Malays. Math. Sci. Soc. (2), **35,** no. 4, (2012), 935–944.

[4] M. Afkhami, Kazem Khashyaramesh, On the Cozero-Divisor graphs of Commutative Rings, Applied Mathematics, **4,** (2013), 979–985, http://dx.doi.org/10.4236/am.2013.47135.

[5] M. Afkhami, K. Khashyarmanesh, The cozero-divisor graph of a commutative ring, Southeast Asian Bull. Math., **35,** (2011), 753–762.

[6] D.M.Cardoso, R.C.Díaz, Oscar Rojo, Distance matrices on the H-join of graphs: a general result and applications, Linear Algebra and its Applications,(2018), https://doi.org/10.1016/j.laa.2018.08.024

[7] Dragoš Cvetković, Peter Rowlinson, Slobodan Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press, 2010.

[8] G. Indulal, I. Gutman, A. Vijayakumar, On distance energy of graphs , MATCH Commun. Math. Comput. Chem., **60,** (2008), 461–472.

[9] H. Kumar, K.L. Patra, B.K. Sahoo, Proper divisor graph of a positive integer, arXiv.2005.O4441v1[math.Co]May 2020.

[10] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, 2008.

[11] M. Young, Adjacency matrices of zero-divisor graphs of integers modulo n, Involve, **8,** no. 5, (2015).

[12] Norman Biggs, Algebraic Graph Theory, Cambridge University Press, 1974.

[13] P.M. Magi, Sr. Magie Jose, Anjaly Kishore, Spectrum of the zero-divisor graph on the ring of integers modulo n, J. Math. Comput. Sci., **10,** (2020), 1643–1666.

[14] P. Sharma, A. Sharma, R.K. Vats, Analysis of Adjacency Matrix and Neighborhood Associated with Zero Divisor Graph of Finite Commutative Rings, International Journal of Computer Applications, **14,** no. 3, (2011), Article 7.

[15] Robert. L. Hemminger, The Group of an X-join of Graphs, Journal of Combinatorial Theory, **5,** (1968), 408–418.

[16] S. Chattopadhyay, K.L. Patra, B.K. Sahoo, Laplacian eigenvalues of the zero divisor graph of the ring Zn, Linear Algebra and its applications, **584,** (2020), 267–286.