International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 613–620

On the commutativity probability in certain finite groups

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(Received October 20, 2023, Revised November 21, 2023,
Accepted November 28, 2023, Published February 12, 2024)

Abstract

The purpose of this paper is to compute the probability $\Pr(G)$ that two elements of the group G, drawn at random with replacement, commute; that is,

$$\Pr(G) = \frac{\text{Number of ordered pairs } (x, y) \in G \times G \text{ such that } xy = yx}{|G \times G| = |G|^2}$$

In particular, we compute $\Pr(G)$ for some groups such as the extraspecial groups of order p^3, p prime, for the permutation groups $G = S_n$ and $G = A_n$, $n \geq 5$, for 10 non-abelian groups of order p^4 and for simple groups of certain type.

1 Introduction.

In this paper, all groups are finite. This notion of probability has been investigated by Gustafson in [11] , where he studied the probability that two group elements commute. In [15], Rusin has considered the probability that two elements of a finite group commute. In particular, he explicitly computed

Key words and phrases: Probability, nilpotent group, commutator subgroup, conjugacy classes.
AMS (MOS) Subject Classifications: 20P05.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

Pr(G) for groups G with $G' \leq Z(G)$ or $G' \cap Z(G) = \{1\}$ where 1 is the identity element of G, and classified the groups for which this probability is greater than $\frac{11}{32}$. In [12], Gallagher has investigated the number of conjugacy classes in a finite group G. For more information about this concept one may refer to [7], [9], [13], [14], [16] and [15] .

2 Notations and preliminaries.

Our notations are fairly standard. If G is a group, then $Z(G)$ denotes the center of G and G' is the commutator subgroup of G or the derived group of G. The number of the conjugacy classes of a finite group of G is denoted by $\left|G^C\right|$. The semidirect product of groups G and H is denoted by $G \rtimes H$.

Definition 2.1. ([10], [13]) Let G be a finite p-group for which $G/Z(G)$ is elementary abelian and $Z(G)$ has order p. Then G is called an extraspecial group.

Example 2.1. The group P^3, the dihedral group D_8, and the quaternion group Q_8 are extraspecial groups.

Definition 2.2. ([10], [13]) Let G be a finite group. Then G is called nilpotent with nilpotency class 2 if and only if $[G, G, G] := [[G, G], G] = \{1\}$ or equivalently $G' \leqslant Z(G)$.

Theorem 2.1. *[15] If G is a p-group with $G' \leqslant Z(G)$, then*

$$\Pr(G) = \frac{1}{|G'|}\left(1 + \sum_{\substack{G'/K \\ cyclic}} \frac{(p-1)\cdot[G' : K]/p}{p^{n(K)}}\right),$$

where $p^{n(K)} = [G/K : Z(G/K)] \geqslant [G' : K]^2$.

Proposition 2.1. *[15] If H is a p-group with $H' \leqslant Z(H)$ and H' cyclic, then $H/Z(H) \cong \prod_i (C_{p^{n_i}} \times C_{p^{n_i}})$ with all $n_i \leq k$, $n_1 = k$, where $p^k = |H'|$. In particular, $[H : Z(H)]$ is square and is at least $|H'|^2$.*

Remark 2.1. *The number of the conjugacy classes of a finite group G is a significant quantity. It is used to measure the probability that two elements commute:*
"Let G be a finite group. Then the commutativity degree of G is $P_r(G) = k/|G|$, where $k = |G^C|$".

Theorem 2.2. *[1] For the symmetric group S_n of cycle length $L = \{0, 2, 3, \dots, n\}$, the number of conjugacy classes uniquely determined by a cycle of length $k \in L$ is M_k, where M_k is the number of solutions of $k_i \in L$ for the inequality $k + \sum_{k \geq k_i \in L} k_i \leq n$. Therefore, $\left|G^C\right| = \sum_{k \in L} M_k$.*

3 Results

In this section, we will show our main results corresponding to Pr(G) for some selective groups G.

Theorem 3.1. *Let G be an extraspecial group of order p^3, p prime. Then*

$$\Pr(G) = \frac{p^2 + p - 1}{p^3}$$

Proof. As $|G| = p^3$, $Z(G) > 1$. This implies that $|G/Z(G)| \leq p^2$. Hence $G/Z(G)$ is abelian which means $(G/Z(G))' = \{1\} = G'Z(G)/Z(G)$. It follows that $G' \leqslant Z(G)$; i.e., G is of nilpotency class 2 . As $Z(G)$ has order p, $|G'| = \mathbb{Z}_p$. The only proper subgroup of G' is $K = \{1\}$, which has index p in G' . Applying Theorem 2.1, we get

$$\Pr(G) = \frac{1}{|G'|}\left(1 + \sum_{\substack{G'/K \\ cylic}} \frac{(p-1)\,[G' : K]\,/p}{p^{n(K)}}\right),$$

where using Proposition 2.1

$$p^{n(K)} = [G/K : Z(G/K)] \geqslant \left[G' : K\right]^2 .$$

Consequently, $\Pr(G) = \frac{1}{p}\left(1 + \frac{(p-1)}{p^2}\right)$, where $G/Z(G) \cong \mathbb{Z}_p^2 = \mathbb{Z}_p \times \mathbb{Z}_p$. □

Proposition 3.1. *For any finite group G, $\Pr(G) \leq \frac{1}{4} + \frac{3}{4}\frac{1}{|G'|}$, where G' is the commutator subgroup of G. If $G = S_n, n \geqslant 5$, then $\Pr(G) \leqslant \frac{1}{16} + \frac{15}{16} \cdot \frac{1}{2n!}$ and if $G = A_n$, $n \geqslant 5$, then $\Pr(G) \leqslant \frac{1}{9} + \frac{8}{9} \cdot \frac{1}{2n!}$.*

Proof. Using character theory ([13], Chapter 5), the degree equation of a finite group G is $|G| = \sum_{i=1}^{k} r_i^2$, where k is the number of conjugacy classes of G, and the r_i are positive integers. More precisely, $[G : G']$ of these are equal to 1 . So

$$|G| = s + \sum_{s+1}^{k} r_i^2$$

where

$$[G : G'] = s \geqslant s + 4(k - s) = 4k - 3s.$$

It follows that $k \leqslant \frac{1}{4}(|G| + 3s)$ and $\Pr(G) = \frac{k}{|G|} \leq \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{|G'|}$.

If $G = S_n$, then $|G| \geqslant 120$ for $n \geqslant 5$ and $k \geqslant 7$ and if $G = A_n$, then $|G| \geqslant 60$ for $n \geqslant 5$ and $k \geqslant 5$. Again from the degree equation of G it follows that $|G| \geqslant [G : G'] + 64\,(k - [G : G'])$ if $G = S_n, n \geqslant 5 = 16k - 15\,[G : G']$. This implies that $k \leqslant \frac{1}{16}\,(|G| + 15\,[G : G'])$ and so $\Pr(G) \leqslant \frac{1}{16} + \frac{15}{16} \cdot \frac{2}{n!}$, where $S'_n = A_n$. Arguing in a similar manner for $G = A_n$, one obtains $P_r(G) \leqslant \frac{1}{9} + \frac{8}{9} \cdot \frac{1}{2n!}$, as $A'{}_n = A_n$ for $n \geqslant 5$ □

Example 3.1. Let $G = S_{12}$. By the above proposition,
$P_r(S_{12}) \leq \frac{1}{16} + \frac{15}{16}.\frac{2}{12!} \approx \frac{1}{16} + \frac{2}{12!}$.

Remark 3.1. *In [1], the following result has been proved for $P_r(S_n)$.*
Let $G = S_n$ be the symmetric group of cycle lengths $L = \{0, 2, 3, ..., n\}$, and let M_k be the number of solutions of $k_i \in L$ for the inequality $k + \sum_{k \geq k_i \in L} k_i \leq n$.

Then $|G^C| = \sum_{k \in L} M_k$ *and* $P_r(G) = \dfrac{\sum_{k \in L} M_k}{|G|}$.

For $G = S_{12}$, the number of solutions has been given for the above inequality using the *GAP* program [5] for computation where $P_r(S_{12})$ was computed as $\frac{77}{12!}$. This is worth comparing with the result obtained in the above example.

Remark 3.2. *[6] There are* 10 *non-abelian groups of order* p^4*; namely,*

1. $F_1 = \left\langle a, b | a^{p^2} = b^p = [a,[a,b]] = [b,[a,b]] = [a,b]^p = e \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.
2. $F_2 = \left\langle a, b | a^{p^2} = b^{p^2} = e, [b,a] = b^p \right\rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{p^2}$.
3. $F_3 = \left\langle a, b | a^{p^3} = b^p = e, [b,a] = a^{p^2} \right\rangle \cong \mathbb{Z}_{p^3} \rtimes \mathbb{Z}_p$. .
4. $F_4 = \langle a, b, c | a^p = b^p = c^p = d^p = [a,c] = [b,c] = [a,d] = [b,d] = [c,d] = e \rangle \cong \mathbb{Z}_p \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)$, *where* $d = [a,b]$.
5. $F_5 = \left\langle a, b, c | a^{p^2} = b^p = c^p = [a,c] = [b,c] = e, [b,a] = a^p \right\rangle \cong \mathbb{Z}_p \times (\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p)$.
6. $F_6 = \left\langle a, b, c | a^{p^2} = b^p = c^p = [a,b] = [a,c] = e, [c,b] = a^p \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.
7. $F_7 = \langle a, b | a^p = b^p = c^p = [a,c]^p = [b,c] = e, [a,[a,c]] = [b,[a,c]] = e \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$. *where* $c = [a,b]$.
8. $F_8 = \left\langle a, b | a^{p^2} = b^p = [a,b]^p = [b,[a,b]] = e, [a,[a,b]] = a^p \right\rangle \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.
9. $F_9 = \left\langle a, b | a^{p^2} = b^p = [a,b]^p = [a,[a,b]] = e, [b,[a,b]] = a^p \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.
10. $F_{10} = \left\langle a, b | a^{p^2} = b^p = [a,b]^p = [a,[a,b]] = e, [b,[a,b]] = a^{2p} \right\rangle \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.

Theorem 3.2. *Let* $G = F_i$, $1 \leq i \leq 6$. *Then* $P_r(G) = \frac{1}{p}(1+\frac{(p-1)}{p^2}) = \frac{p^2+p-1}{p^3}$.

Proof. As $|G'| = p$, p prime, and G is nilpotent of nilpotency class 2 [13], the only proper subgroup of G' is $\{1\}$ which has index p. Applying Theorem 2.1 $P_r(G) = \frac{1}{p}(1+\sum_{G'/K} \frac{(p-1)[G':K]/p}{p^{n(K)}})$, $p^{n(K)} \geq |[G':K]|^2 = p^2$ and $G/Z(G) \cong \mathbb{Z}_{p^{2n}}$ y Proposition 2.1. In our case, $n = 2$. ☐

Theorem 3.3. *[3] If G is a finite group of nilpotency class c, Then the number of the conjugacy classes of G is $|G^C| \leq u$, where $u = |G| - r$ and $r = \frac{|G|-|Z(G)|}{c}$.*

Corollary 3.1. *Let G be as in the above theorem. Then $P_r(G) \leq \frac{u}{|G|}$.*

The next remarks can be obtained by Corollary 3.1 to find an upper bound of Pr(G) for all the groups $F_i, i = 1, 2, ..., 10$.

Remark 3.3. *Let $G = F_i$, $1 \leq i \leq 6$. Then $P_r(G) \leq \frac{1}{2} - \frac{1}{2p^2}$.*

Proof. These groups are nilpotent of nilpotency class $c = 2$, $|Z(G)| = p^2$ [6]. Then $r = \frac{p^4-p^2}{2} = u$. Hence $P_r(G) \leq (\frac{p^4-p^2}{2})/p^4 = \frac{1}{2} - \frac{1}{2p^2}$. □

Example 3.2. For $G = F_4$, $c = 2$, $|Z(G)| = 25$. By the previous remark, $P_r(G) \leq \frac{1}{2} - \frac{1}{50} = \frac{12}{25}$. Using Theorem 3.2, $P_r(G) = \frac{1}{5} + \frac{4}{125} = \frac{29}{125} \approx \frac{6}{25}$.

Remark 3.4. *Let $G = F_i$, $8 \leq i \leq 10$. Then $P_r(G) \leq \frac{2}{3} - \frac{1}{3p^3}$, p is prime.*

Proof. These groups are nilpotent of nilpotency class $c = 3$, $|Z(G)| = p$ [6]. Then $r = \frac{|G|-|Z(G)|}{c} = \frac{p^4-p}{3}$ and $u = |G| - r = p^4 - \frac{p^4-p}{3}$. Hence, by Corollary 3.1, $P_r(G) \leq \frac{u}{|G|}$. So $P_r(G) \leq (p^4 - \frac{p^4-p}{3})/p^4 = 1 - \frac{p^4-p}{3p^4} = \frac{2p^4-p}{3p^4} = \frac{2}{3} - \frac{1}{3p^3}$. □

Theorem 3.4. *Let G be a nilpotent group with $G = H_{p_1} \times H_{p_2} \times ... \times H_{p_r}$, p-groups with nilpotency class 2, and the commutator subgroups H'_{p_i} of H_{p_i} are cyclic p-groups C_{p_i}, p_i are primes, $i = 1, ..., r$. Then*

$$P_r(G) = \prod_{i=1}^{r} \frac{1}{p_i} \left(1 + \frac{1}{p_i^{2n_i}}\right).$$

Proof. As noted in [11], we use the general formula $P_r(H \times K) = P_r(H) \times P_r(M)$, where H and M are two finite groups of coprime order. The only subgroup of C_{p_i} for $i = 1, ..., r$ is $\{1\}$ and $C_{p_i/\{1\}}$ is cyclic. Using Theorem 2.1 and Proposition 2.1, one obtains

$$P_r(G) = \prod \frac{1}{|H'_{p_i}|}(1 + \sum_{H'_{p_i}/K} \frac{(p-1).[H'_{p_i} : K]/p}{p^{n_i(K)}}) = P_r(G) =$$

$$\frac{1}{|H_{p_1}|}.(1 + \frac{1}{p_1^{2n_1}}).\frac{1}{|H_{p_2}|}.(1 + \frac{1}{p_2^{2n_2}})...\frac{1}{|H_{p_r}|}.(1 + \frac{1}{p_r^{2n_r}}),$$

for some n_i where $H_{p_i/Z(H_{p_i})} \cong C_{p_i}^{2n_i}$ for some n_i, $[H_{p_i} : Z(H_{p_i})]$ is a square, and is at least $|H'_{p_i}|^2$ and $p^{n_i(K)} \geq [H'_{p_i} : K]$ where $H'_{p_i/K}$ is cyclic. □

Corollary 3.2. *The above theorems can be applied to the simple Janko groups* J_1, J_2 *and the Mathieu group* M_{12} *of order* $|J_1| = 2^3.3.5.7.11.19$*,* $|J_2| = 2^7.3^3.5^2.7$ *and* $|M_{12}| = 2^4.3^2.5.11$ *and their centers are of order 1 and their subgroups* $J_1' = J_1, J_2' = J_2$ *and* $M_{12}' = M_{12}$*. The computations are easy and so are omitted.*

Acknowledgment

The author is grateful to the Public Authority for Applied Education and Training for supporting this project No BE-22-09, and to Professor Mashhour Bani-Ata for his helpful discussion.

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