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A Characterization of Partition-Good and Partition-Wonderful Complete Multipartite Graphs

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Abstract

A simple graph G on n vertices is **partition-good** if and only if for all pairs (a, b) of positive integers such that $a+b = n$, $V(G)$ can be partitioned into sets A, B satisfying: $|A| = a, |B| = b$, and $G[A], G[B]$ are connected. G is **partition-wonderful** if and only if either (i) $n = 1$, or (ii) $n > 1$, G is connected, and of all pairs (a, b) of positive integers such that $a + b = n$, $V(G)$ can be partitioned into sets A, B satisfying: $|A| = a, |B| = b$, and $G[A], G[B]$ are partition-wonderful. We characterize the partition-good and the partition-wonderful among the complete multipartite graphs, and, along the way, prove some elementary results about these graph properties.

1 Introduction

All following graphs are finite and simple. A subgraph of a graph G is *rainbow* with respect to a coloring of $E(G)$ if and only if no color appears on more than one edge of the subgraph. It is well known that if G is a connected graph on n vertices, and $E(G)$ is colored with n or more colors, then there

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must be a *rainbow* cycle in G with respect to the coloring. (See [1] and [2].) On the other hand, each such G can be edge colored with exactly $n-1$ colors appearing such that no cycle in G is rainbow. In conformity with [1],[3],[4], and [5], we will call such a coloring a JL coloring of G.

There is a straightforward way to obtain JL colorings of a connected graph G. You partition $V(G)$ into non-empty sets A, B such that the induced subgraphs $G[A], G[B]$ are connected. (One way to do this: take a spanning tree in G and remove one edge from it.) Color all of the edges in the edge cut $[A, B]$ with one color that will never be used again. Iterate the process on $G[A], G[B]$ and continue until all of the edges of G are colored.

The main result of [5] is that for every simple finite connected graph G, every JL coloring is obtainable by an instance of the process just described. It is an easy corollary of this result that if G is connected, on n vertices, then each JL coloring of G is the restriction to $E(G)$ of a JL coloring of the complete graph K on $V(G)$. To put it another way: each JL coloring of G can be extended to a JL coloring of K. This raises the question (first raised by a comment from Luc Teirlinck, in seminar, for which we thank him): Which connected graphs G have the property that for every JL coloring of K_n where $n = |V(G)|$, there is a subgraph $\tilde{G}$ of K_n isomorphic to G such that the restriction of the coloring of K_n to the edges of $\tilde{G}$ gives a JL coloring of $\tilde{G}$?

This property is partition-wonderfulness, defined recursively and quite differently in the next section. A weaker property, partition-goodness, arises from contemplation of the first stage of the JL coloring construction process described above: For which connected graphs G on n vertices can the cardinalities $a = |A|$ and $b = |B|$ in the first stage of the process be arbitrarily specified?

2 Preliminary Definitions

Definition 2.1. *Let G be a simple graph. A partition of $V(G)$ into subsets A and B is **good** if and only if $G[A]$ and $G[B]$ are connected.*

Definition 2.2. *A graph G is **partition-good** if and only if for any pair (a, b) of positive integers such that $a + b = |V(G)|$, there is a good partition of $V(G)$ into A and B such that $|A| = a$ and $|B| = b$.*

Note the following two special cases of partition-good graphs:

i. $\overline{K_2}$

ii. $K_1 + K_2$

Claim: $\overline{K_2}$ and $K_1 + K_2$ are the only disconnected partition-good graphs.

Proof:
It is clear that $\overline{K_2}$ and $K_1 + K_2$ are partition-good.

Suppose that G is partition-good and disconnected. Let $c = \max\{|V(H)| : H$ is a connected component of $G\}$.

Claim 1: G has exactly two components.

Because G is partition-good and $0 < c < |V(G)| = n$, there is a partition of $V(G)$ into sets C and B, $|C| = c$ and $|B| = n - c$, such that both $G[C]$ and $G[B]$ are connected. Then each of these subgraphs must be subgraphs of connected components of G. By the definition of c, $G[C]$ must be one of those connected components; then $G[B] = G - G[C]$ is a connected component of G. Thus G has two connected components.

Claim: $|V(G)| = c + 1$

Since G has more than one component, it must be the case that $|V(G)| \geq c+1$. Suppose, $|V(G)| > c+1$. By assumption, G is partition-good, so there must exist $A \subseteq V(G)$ such that $a = |A| = c + 1$, and $G[A]$ is connected. This is not possible because G has no connected subgraph of order greater than c. Therefore, $|V(G)| = c + 1$. Since G only has two components, one component must be an isolate.

Claim: $c \in \{1, 2\}$

Let H_s denote the component of G that is an isolate and H_c denote G's other component of order c. Suppose that $c > 2$ and consider $a = 2$. If $G[A]$ is to be connected, then both vertices must lie in H_c. Since both vertices of A lie in H_c, $G[V(G) \setminus A]$ must have one vertex in H_s and $c - 2$ vertices in H_c. Thus, $G[V(G) \setminus A]$ is not connected which contradicts the assumption that G is partition-good. Therefore, either $c = 2$ and $G = K_1 + K_2$ or $c = 1$ and $G = \overline{K_2}$. □

Definition 2.3. *Let G be a simple graph. K_1 is **partition-wonderful**. If $|V(G)| > 1$, then G is **partition-wonderful** if and only if the following hold:*

1. *G is connected.*
2. *For every pair (a, b) of positive integers such that $a + b = |V(G)|$ there are disjoint sets $A, B \subseteq V(G)$ such that $|A| = a$, $|B| = b$, and $G[A]$ and $G[B]$ are partition-wonderful.*

Lemma 2.4. *Every path is partition-wonderful.*

Proof:
The proof that P_n, the path on n vertices, is partition-wonderful will be by induction on n.

$P_1 = K_1$, which is partition-wonderful by definition.

Suppose that $n > 1$ and a, b are positive integers such that $a + b = n$. Let A consist of a consecutive vertices along the path $P = P_n$ starting from one end, and let $B = V(P) \setminus A$. Clearly, $P[A] = P_a$ and $P[B] = P_b$. Since $a, b < n$, P_a and P_b are partition-wonderful, by the induction hypothesis. Since a, b were arbitrarily chosen, it follows that P is partition-wonderful. $\square$

3 Preliminary Results

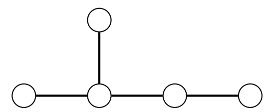
Proposition 3.1. *Let G be a simple graph. If G has a partition-wonderful spanning subgraph, then G is partition-wonderful.*

Proof:
Proof by induction on $n = |V(G)|$: Suppose that G has a partition-wonderful spanning subgraph H. Then for all pairs (a, b) such that a, b are positive integers and $a + b = |V(G)| = |V(H)|$, there exists a partition of $V(H)$ into A and B such that $|A| = a$, $|B| = b$, and $H[A]$ and $H[B]$ are partition-wonderful.

Then $G[A], G[B]$ each has a partition-wonderful spanning subgraph and is therefore partition-wonderful by the induction hypothesis. Since this holds for all such pairs (a, b), it follows that G is partition-wonderful. $\square$

Corollary 3.2. *Let G be a simple graph. If G has a Hamilton path, then G is partition-wonderful.*

The converse of Corollary 3.2 is false. Consider the following:



G

G is clearly partition-wonderful but has no Hamilton path.

Lemma 3.3. *If G is partition-wonderful, then G is partition-good.*

Proof: Clear, by the definition of partition-wonderful.

4 Results for Complete Multipartite Graphs

Theorem 4.1. *Let $G = K_{1,t}$. G is partition-good if and only if $t \in \{1, 2\}$.*

Proof:
If $t = 1$, $G = K_{1,1} = P_2$ and if $t = 2$, $G = K_{1,2} = P_3$. Since all paths are partition-good, G is partition-good both when $t = 1$ and $t = 2$. Consider $t \geq 3$. $G = K_{1,t}$, so $|V(G)| = t + 1$. Let v_0 denote the central vertex and v_i, $i \in \{1, ..., t\}$ be the leaves of G. Let $a = |A|$ and $b = |B|$ where $a = 2$ and $b = t - 1$. Then, to be connected, $G[A]$ must be a P_2 formed by taking A to be v_0 and any single v_i where $i \in \{1, ..., t\}$. Since $B = V(G) \setminus A$, $G[B]$ must be the remaining $t - 1$ vertices that were leaves in G forming $\bar{K}_{t-1}$. Thus, $G[B]$ is not connected and $G = K_{1,t}$ is not partition-good. □

Corollary 4.2. *Let $G = K_{1,t}$. G is partition-wonderful if and only if $t \in \{1, 2\}$.*

Theorem 4.3. *Let $G = K_{p_1,p_2}$. G is partition-good for all $2 \leq p_1 \leq p_2$.*

Proof:
Let $G = K_{p_1,p_2}$ such that $2 \leq p_1 \leq p_2$ and $p_1 + p_2 = n$. Let a, b, A, and B be as in Definition 2.2 with $a \leq b$, and let the parts of G be P_1 and P_2 such that $|P_1| = p_1$ and $|P_2| = p_2$. If $a = 1$, let A be any vertex from P_1; then $G[A]$ is an isolate. Then $G[B] = K_{p_1-1,p_2}$, and $G[A], G[B]$ are both connected. For $a > 1$, choose A, B as follows. Let A consist of a single vertex from P_1 and $t = a - 1$ vertices from P_2 and B consist of $p_1 - 1$ vertices from P_1 and $s = p_2 - t$ vertices from P_2. Then, $G[A]$ is a $K_{1,t}$ and $G[B]$ is a $K_{p_1-1,s}$, both of which are connected. □

Theorem 4.4. *Let $G = K_{p_1,p_2}$ where $2 \leq p_1 \leq p_2$. G is partition-wonderful if and only if $p_2 \leq p_1 + 1$.*

Proof:
Let P_1, P_2 be the parts of G where $|P_i| = p_i$ for $i \in \{1, 2\}$. If $p_2 \leq p_1 + 1$, then $p_2 \in \{p_1, p_1 + 1\}$. Then G has a Hamilton path, and, by Corollary 3.2, is partition-wonderful.

Suppose that $p_2 \geq p_1 + 2$. We will show that G is not partition-wonderful by induction on the order of G, $n = p_1 + p_2$. Minimally, $G = K_{2,4}$ as $p_1 \geq 2$. We shall show that $G = K_{2,p_2}$ is not partition-wonderful for all $p_2 \geq 4$. Let $a = 2$ and $b = (p_2 + 2) - 2 = p_2$, and suppose that $V(G)$ is partitioned into A and B such that $|A| = 2, |B| = p_2$, and $G[A], G[B]$ are connected. Then $A = \{x, y\}$ for some $x \in P_1,\ y \in P_2$, so B consists of one vertex from P_1 and $p_2 - 1 \geq 3$ vertices from P_2. Then $G[B] = K_{1,p_2-1}$, which is not partition-wonderful either by Corollary 4.2 or by Lemma 3.3. Thus, G is not partition-wonderful.

Now suppose that $3 \leq p_1$ and $p_1 + 2 \leq p_2$, so $2 \leq p_1 - 1 \leq (p_2 - 1) - 2$. Therefore, again taking $a = 2, b = p_1 + p_2 - 2$ and A and B as usual, $G[B] = K_{p_1-1,p_2-1}$ is not partition-wonderful, by the induction hypothesis. Thus $G = K_{p_1,p_2}$ is not partition-wonderful, because no partition A, B of $V(G)$ can be found such that $|A| = 2, |B| = n - 2$, and both $G[A]$ and $G[B]$ are partition-wonderful. □

Theorem 4.5. *If $r > 2$ and $p_1, ..., p_r$ are positive integers, then $G = K_{p_1,...,p_r}$ is partition-good.*

Proof:
Without loss of generality, $p_1 \leq ... \leq p_r$. Let $P_1, ..., P_r$ be the parts of $G = K_{p_1,...,p_r}$ so that $|P_j| = p_j$ We will proceed by induction on $|V(G)| = n = \sum_{j=1}^{r} p_j$. The smallest n can be is $n = 3$ with $r = 3$ and $p_1 = p_2 = p_3 = 1$. Then, $G = K_3$, which is clearly partition-good.
Suppose that $n > 3$ and that $p_r > 1$ (as $p_r = 1$ gives $G = K_r$ which is partition-good for all r). We may, also, assume that $b \geq n - b = a > 1$, since the case $a = 1, b = n - 1$ is easily disposed of. Let $x \in P_r$. Consider $H = G - x = K_{p_1,...,p_{r-1}}$. By the induction hypothesis, we can partition $V(H) = V(G) \setminus \{x\}$ into sets A, B' such that $|A| = a, |B'| = b - 1$, and $H[A], H[B']$ are connected. Take $B = B' \cup \{x\}$. Since $H \subseteq G$, $G[A] = H[A]$ and the only way that $G[B]$ is not connected is if $B' \subseteq P_r \setminus \{x\}$. However,

B' is connected, so, in that case, $|B'| = 1 = b - 1$. Then, $a = b = 2$ and $n = 4$ since $1 < a \leq b$. Since $r \geq 3$, $G = K_{1,1,2}$ which has a Hamilton path, and, thus, is partition-good by Corollary 3.2. ☐

Lemma 4.6. *Suppose that $r > 2$ and $p_1, ..., p_r$ are integers satisfying $1 \leq p_1 \leq ... \leq p_r$. Then $G = K_{p_1,...,p_r}$ has a Hamilton path if and only if*

$$p_r \leq \left(\sum_{j=1}^{r-1} p_j\right) + 1$$

.

Proof:
Let the parts of G be $P_1, ..., P_r$ of cardinalities $p_1, ..., p_r$, respectively. Let $n = |V(G)| = \sum_{j=1}^{r} p_j$.

If G has a Hamilton path Q, then $|E(Q)| = n - 1$. At most two vertices of P_r have degree 1 on Q. The others have degree 2 on Q and each edge incident to a vertex in P_r has its other end in $\bigcup_{j=1}^{r-1} P_j$. Therefore,

$$\begin{aligned} n - 1 = |E(Q)| &\geq |\{\text{edges of } Q \text{ incident to vertices in } P_r\}| \\ &\geq 2 + 2(p_r - 2) = 2p_r - 2 \end{aligned}$$

Then,

$$\begin{aligned} 2p_r \leq n + 1 = p_r + \left(\sum_{j=1}^{r-1} p_j\right) + 1 \\ \implies p_r \leq \left(\sum_{j=1}^{r-1} p_j\right) + 1. \end{aligned}$$

Now, suppose that $p_r \leq \left(\sum_{j=1}^{r-1} p_j\right) + 1$. If $p_r = 1$, then $G = K_r$ which has a Hamilton cycle, and thus a Hamilton path. If $p_r \in \left\{\left(\sum_{j=1}^{r-1} p_j\right), \left(\sum_{j=1}^{r-1} p_j\right) + 1\right\}$, then the complete bipartite graph with bipartition P_r, $\bigcup_{j=1}^{r-1} P_j$, which is a spanning subgraph of G, has a Hamilton path, and so, consequently, G has one as well.

Therefore, we can assume that $1 < p_r < \left(\sum_{j=1}^{r-1} p_j\right)$. We proceed by induction on n. The smallest value of n possible when $1 < p_r < \left(\sum_{j=1}^{r-1} p_j\right)$ is 5, and the graph is $K_{1,2,2}$, which has a Hamilton cycle. Now, suppose that $n > 5$. Let $x \in P_r$ and $G' = G - x$. Then $G' = K_{q_1,\ldots,q_r}$ with $1 \leq q_1 \leq \ldots \leq q_r$, where either

1. $q_r = p_r - 1$ and $q_j = p_j$ for $j = 1, \ldots, r-1$, or
2. $q_r = p_{r-1} = p_r$, $q_j = p_r - 1$ for some $j \in \{1, \ldots, r-1\}$ such that $p_j = p_r$ and $q_i = p_i$ for all $i \in \{1, \ldots, r-1\} \setminus \{j\}$.

In case 1, either $q_r = 1$, in which case $G' = K_r$, or $1 < q_r = p_r - 1 < \left(\sum_{j=1}^{r-1} p_j\right) - 1 = \left(\sum_{j=1}^{r-1} q_j\right) - 1$.

In either subcase, G' has a Hamilton path Q': in the second subcase, the existence of Q' is implied by the induction hypothesis.

In case 2, we have

$$\begin{aligned}\sum_{j=1}^{r-1} q_j &= \sum_{j=1}^{r-1} p_j - 1\\ &= p_{r-1} + \sum_{j=1}^{r-2} p_j - 1\\ &= p_r + \sum_{j=1}^{r-2} p_j - 1 \geq p_r = q_r > 1.\end{aligned}$$

When $p_r + \sum_{j=1}^{r-2} p_j - 1 > p_r$, G' has a Hamilton path Q' by the induction hypothesis. In case of equality (which is possible only when $r = 3, p_1 = 1$), the existence of Q' follows from previous arguments.

If Q' contains an edge yz with $y \in P_i, z \in P_j, 1 \leq i < j < r$, then we can obtain a Hamilton path in G by replacing yz by two edges yx and xz. If no such edge yz exists, then every edge of Q' has one end in $P_r \setminus \{x\}$. But then,

$$\begin{aligned} |E(Q')| = n - 2 = p_r + \sum_{j=1}^{r-1} p_j - 2 \\ \leq 2(p_r - 1) \\ \implies p_r \geq \sum_{j=1}^{r-1} p_j \end{aligned}$$

contrary to supposition. □

Remark: The proof of Lemma 4.6 can be modified to prove the following, a generalization of one of the main results of [6].
If $r \geq 3$ and integers $p_1, ..., p_r$ satisfy $1 \leq p_1 \leq ... \leq p_r$ then $K_{p_1,...,p_r}$ has a Hamilton cycle if and only if $p_r \leq \sum_{j=1}^{r-1} p_j$.

We strongly suspect that this is well known; what's more, Lemma 4.6 can be deduced easily from it.

Theorem 4.7. *Let G be a complete multipartite graph with r partite sets of sizes $1 \leq p_1 \leq p_2 \leq ... \leq p_{r-1} \leq p_r$ where $r > 2$. Then G is partition-wonderful if and only if*

$$p_r \leq \left(\sum_{j=1}^{r-1} p_j\right) + 1.$$

Proof:
Suppose that $p_r \leq \left(\sum_{j=1}^{r-1} p_j\right) + 1$; then, by Lemma 4.6, G has a Hamilton path. Therefore, by Corollary 3.2, G is partition-wonderful.

Now, suppose that $p_r > \left(\sum_{j=1}^{r-1} p_j\right) + 1$. Note that this implies that $p_r - 1 > p_{r-1} + 1$. We will proceed by induction on $|V(G)| = n = \sum_{j=1}^{r} p_j$. Minimally, $G = K_{1,1,4}$. Let a, b, A, B be as in Definition 2.3, and consider $a = 2, b = 4$. It must be that $G[A] = K_2$ where at least one endpoint lies in either P_1 or P_2. Suppose that $A = \{x, y\}$, and, without loss of generality, suppose that

$x \in P_1$. Then $y \in P_2$ or $y \in P_3$. If $y \in P_2$, then $B = P_3$ and $G[B]$ is not connected. If $y \in P_3$, $G[B] = K_{1,3}$ which is not partition-wonderful (by Corollary 4.2). Thus $G = K_{1,1,4}$ is not partition-wonderful.

Assume that $n > 6$ and suppose that all complete multipartite graphs such that $r > 2$ and $p_r > \left(\sum_{j=1}^{r-1} p_j\right) + 1$ with order less than n are not partition-wonderful. Suppose that $a = 2$ and $b = n - 2$. Then, either $A = \{x, y\}$ such that $x \notin P_r$, $y \notin P_r$ or $A = \{x, y\}$ and either $x \in P_r$ or $y \in P_r$. In either case, $G[A] = K_2$.

Case 1: $A = \{x, y\}$ such that $x \notin P_r, y \notin P_r$

$G[B] = K_{p_1',\dots p_{r-1}',p_r'}$ where $p_j' = p_j$ (for $1 \leq j \leq r$) if $x \notin P_j$ and $y \notin P_j$ and $p_j' = p_j - 1$ if $x \in P_j$ or $y \in P_j$. Then,

$$\sum_{j=1}^{r-1} p_j' = \left(\sum_{j=1}^{r-1} p_j\right) - 2 < \left(\sum_{j=1}^{r-1} p_j\right) + 1 < p_r = p_r'.$$

Clearly $|G[B]| < n$, so by the induction hypothesis, $G[B]$ is not partition-wonderful.

Case 2: $A = \{x, y\}$ such that $x \in P_r$ or $y \in P_r$
Without loss of generality, assume that $x \in P_r$. Then $G[B] = K_{p_1',\dots p_{r-1}',p_r'}$ where for $1 \leq j \leq r$, $p_j' = p_j$ if $x, y \notin P_j$ and $p_j' = p_j - 1$ if $x \in P_j$ or $y \in P_j$. Note that $p_r - 1 > p_{r-1} + 1$ implies that $p_r' = p_r - 1$ is the largest of the p_j'. By assumption, $|B| = b = n - 2$, and

$$p_r > \left(\sum_{j=1}^{r-1} p_j\right) + 1 = \left(\left(\sum_{j=1}^{r-1} p_j'\right) + 1\right) + 1$$

since $y \in P_j$ for some $1 \leq j \leq r - 1$. However, $p_r = p_r' + 1$, so

$$p_r' + 1 > \left(\left(\sum_{j=1}^{r-1} p_j'\right) + 1\right) + 1$$

$$\Longrightarrow p_r' > \left(\sum_{j=1}^{r-1} p_j'\right) + 1.$$

Then, by the induction hypothesis, $G[B]$ is not partition-wonderful. Therefore, when $a = |A| = 2$, there is no partition-wonderful $G[B]$. Therefore, $G = K_{p_1,\ldots,p_r}$ is partition-wonderful if and only if $p_r \leq \left(\sum_{j=1}^{r-1} p_j\right) + 1$. $\square$

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