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The $f(x), g(x)$-clean property of rings

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Abstract

In this paper, we introduce a new property of rings called the $f(x), g(x)$-clean property, which is a generalization of the $g(x)$-clean property. We prove that a commutative ring Q is feebly clean if and only if it is $f(x), f(x)$-clean, where $f(x)$ is a suitable quadratic polynomial. We prove additional results of this property and include examples for clarity.

1 Introduction

In this paper, all rings are nontrivial and have identity elements. We Use $C(Q), N(Q), J(Q)$ and $U(Q)$ to denote the center, the nilradical, the Jacobson radical, and the unit elements of a ring Q respectively. In 1977, Nicholson [1] defined a property of rings called the clean property: A ring Q is said to be clean if we can write every element of Q as a sum of a unit element and an idempotent element. This definition was introduced by him to study the exchange property of rings. In particular, he showed that clean rings are exchange rings.

Many generalizations of clean rings have been introduced in the past five decades. Let Q be a ring and $g(x) \in C(Q)[x]$ be a polynomial with central

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coefficients. Camilo and Simón [2] called Q a $g(x)$-clean ring, if it is possible to write every $q \in Q$ as $q = v + b$, where $v \in U(Q)$ is an arbitrary unit element and $b \in Q$ is an arbitrary root of $g(x)$; i.e., $g(b) = 0$. Later, Fan and Yang [3] proved that a ring Q is clean if and only if it is $(x - r)(x - s)$-clean, where $r, s \in C(Q)$ with $s - r \in U(Q)$.

Another generalization of clean rings was defined and explored by Arora and Kundu [4]. A ring Q is called feebly clean if we can write every $q \in Q$ as $q = v + b - c$, where $v \in U(Q)$ and $b, c \in Q$ are orthogonal idempotents; i.e., $b^2 = b, c^2 = c$ and $bc = cb = 0$. Among many results, they showed that the ring of complex-valued continuous functions $C(Y, \mathbb{C})$ of a topological space Y is clean if and only if it is feebly clean.

In this paper, we continue this line of research by introducing a novel generalization of the clean property of rings called the $f(x), g(x)$-clean property. We prove certain relations between this newly defined generalization and the ones we mentioned earlier.

2 Main results

Definition 2.1. *Let Q be a ring and let $f(x), g(x) \in C(Q)[x]$ be two polynomials with central coefficients. Q is said said to be $f(x), g(x)$-clean if we can write every $q \in Q$ as $q = v + \eta + \theta$, where $v \in U(Q)$ and $\eta, \theta \in Q$ are roots of $f(x)$ and $g(x)$ respectively; i.e., $f(\eta) = g(\theta) = 0$.*

It is a simple exercise to show that if a ring Q is $g(x)$-clean for some $g(x) \in C(Q)[x]$, then it is also $f(x), g(x)$-clean for any $f(x) \in C(Q)[x]$ that has at least one root in Q. The following example shows that the converse is not true in general. That is, the $f(x), g(x)$-clean property is a nontrivial generalization to the $g(x)$-clean property of [2] in the sense that there are rings which are $f(x), g(x)$-clean but are neither $f(x)$-clean nor $g(x)$-clean.

Example 2.2. *For any $n \geq 2$, the ring $Q_n = UT_n(\mathbb{Z}_6)$ of $n \times n$ upper triangular matrices with $\mathbb{Z}_6$ entries is $(x^2-2x), (x^2-3x)$-clean but it is neither $(x^2 - 2x)$-clean nor $(x^2 - 3x)$-clean. The ring $\mathbb{Z}_6$ itself is $(x^2 - 2x), (x^2 - 3x)$-clean but it is neither $(x^2 - 2x)$-clean nor $(x^2 - 3x)$-clean. Therefore, Q_n has the aforementioned properties by Proposition 2.8 and [3, Proposition 2.8].*

In Definition 2.1, we do not rule out the scenario where $f(x) = g(x)$. A ring in which each element can be expressed as the sum of a unit and two roots of $f(x)$ is referred to as an $f(x), f(x)$-clean ring.

Theorem 2.3. *Let Q be a commutative ring and let $r, s \in Q$ such that $s-r \in U(Q)$. Set $f(x) = (x-r)(x-s)$. Then, Q is feebly clean if and only if Q is $f(x), f(x)$-clean.*

Proof. Assume that Q is feebly clean and let $q \in Q$. Then $q = v + b - c$ where $v \in U(Q)$ and b, c are orthogonal idempotents. Since c satisfies $c^2 = c$, $-c$ satisfies $c^2 + c = 0$ and so Q is $(x^2 - x), (x^2 + x)$-clean. As $s - r \in U(Q)$, we can write $\frac{q-r-s}{s-r} = u + \eta + \theta$ where $u \in U(Q)$ and $\eta, \theta \in Q$ such that $\eta^2 - \eta = \theta^2 + \theta = 0$. This gives $q = (s-r)u + (s-r)\eta + r + (s-r)\theta + s$ with $(s-r)u \in U(Q)$. Note that $f([s-r]\eta+r) = ([s-r]\eta+r-r)([s-r]\eta+r-s) = [s-r]^2\eta^2 - [s-r]^2\eta = 0$ because $\eta^2 = \eta$. So $[s-r]\eta+r$ is a root of $f(x)$. Also, $f([s-r]\theta+s) = ([s-r]\theta+s-r)([s-r]\theta+s-s) = [s-r]^2\theta^2 + [s-r]^2\theta = 0$ because $\theta^2 = -\theta$. Hence $[s-r]\theta + s$ is also a root of $f(x)$. Therefore, Q is $f(x), f(x)$-clean ring. Conversely, assume that Q is $f(x), f(x)$-clean and let $q \in Q$. Then $(s-r)q + r + s = u + \eta + \theta$ where $u \in U(Q)$ and $\eta, \theta \in Q$ such that $f(\eta) = f(\theta) = 0$. As a result, $q = \frac{u}{s-r} + \frac{\eta-r}{s-r} + \frac{\theta-s}{s-r}$ with $\frac{u}{s-r} \in U(Q)$. Note that $\left(\frac{\eta-r}{s-r}\right)^2 = \frac{(\eta-r)(\eta-s+s-r)}{(s-r)^2} = \frac{(\eta-r)(s-r)}{(s-r)^2} = \frac{\eta-r}{s-r}$. Also, $\left(\frac{\theta-s}{s-r}\right)^2 = \frac{(\theta-s)(\theta-r+r-s)}{(s-r)^2} = -\frac{(\theta-s)(s-r)}{(s-r)^2} = -\frac{\theta-s}{s-r}$ and this shows that Q is $(x^2-x), (x^2+x)$-clean. Consequently, $q = v + b + c$ where $v \in U(Q)$ and $b, c \in Q$ such that $b^2 - b = c^2 + c = 0$ and this allows us to write $q = v + b - (-c)$ where $-c$ is now an idempotent. Set $-c = d$ and note that $q = v + b - d = v + b(1-d) - d(1-b)$ where now $b(1-d)$ and $d(1-b)$ are orthogonal idempotents. This proves that Q is a feebly clean ring. □

Lemma 2.4. *Let Q be a commutative ring, $r \in Q$, $s \in rQ$ and $m, n \in \mathbb{N}$. If Q is $(x^2 - r^m x), (x^2 - s^n x)$-clean, then $r \in U(Q)$.*

Proof. On the contrary, suppose that $r \notin U(Q)$ and consider first the case when $m \leq n$. As $r \notin U(Q)$, $Q/r^mQ \neq 0$. By the $(x^2 - r^m x), (x^2 - s^n x)$-clean property of Q, we can write $r^m = v + \eta + \theta$ where $v \in U(Q)$ and $\eta, \theta \in Q$ such that $\eta^2 - r^m\eta = \theta^2 - s^n\theta = 0$. Passing to Q/r^mQ, we get $r^m + r^mQ = v + \eta + \theta + r^mQ$ so $v + \eta + r^mQ = -\theta + r^mQ$. Squaring the last equation gives $v^2 + 2v\eta + \eta^2 + r^mQ = \theta^2 + r^mQ$. The last equation can be written as $v^2 + 2v\eta + r^m\eta + r^mQ = s^n\theta + r^mQ$ which simplifies to $v^2 + 2v\eta + r^mQ = (rt)^n\theta + r^mQ$ for some $t \in Q$. It follows that, $v^2 + 2v\eta + r^mQ = r^m r^{n-m} t^n \theta + r^mQ$ which implies that $v^4 + r^mQ = 4v^2\eta^2 + r^mQ = 4v^2 r^m \eta + r^mQ = r^mQ$. That is, $v^4 (\in U(Q))$ is in r^mQ, a contradiction. For the case $n \leq m$, we pass to Q/r^nQ and use a similar argument to reach a contradiction. Therefore, $r \in U(Q)$. □

Proposition 2.5. *If a commutative ring Q is $(x^2 - r^m x), (x^2 - r^m x)$-clean for some $r \in Q$, then Q is feebly clean.*

Proof. By Lemma 2.4, $r \in U(Q)$ so also $r^m \in U(Q)$. Now, $x^2 - r^m x = x(x - r^m)$ and the second implication of Theorem 2.3 finishes the proof as now all the assumptions in the statement of this theorem are fulfilled. □

The next lemma is well known.

Lemma 2.6. *Let I be an ideal of a ring Q such that $I \subseteq J(Q)$. Then $v + I$ is a unit of Q/I if and only if v is a unit of Q.*

Theorem 2.7. *Let Q be a commutative ring and let $f(x), g(x) \in Q[x]$. If $Z(f), Z(g) \subseteq J(Q)$, then Q is $f(x), g(x)$-clean if and only if it is $f(x)$-clean and $g(x)$-clean, where $Z(f), Z(g)$ are the zero sets of $f(x)$ and $g(x)$ respectively.*

Proof. If Q is either $f(x)$-clean or $g(x)$-clean, then it is $f(x), g(x)$-clean (see the comment after Definition 2.1). Conversely, assume that Q is $f(x), g(x)$-clean, and let $q \in Q$ be an arbitrary element. We can write $q = v + \eta + \theta$, where $v \in U(Q)$ and $\eta, \theta \in Q$ such that $f(\eta) = g(\theta) = 0$. Let I_f be the ideal generated by the roots of $f(x)$ and let I_g be the ideal generated by the roots of $g(x)$. Passing to Q/I_g, we have $q + I_g = v + \eta + \theta + I_g = v + \eta + I_g$. Consequently, $v + I_g = q - \eta + I_g$ is a unit in Q/I_g. By hypothesis, we have $I_g \subseteq J(Q)$. Therefore, by Lemma 2.6, $q - \eta \in U(Q)$. This allows us to write $q = (q - \eta) + \eta$, where now $q - \eta$ is a unit. Therefore, q is $f(x)$-clean element so Q is a $f(x)$-clean ring. The proof of the $g(x)$-clean property of Q is achieved by passing to Q/I_f and using the same argument above. □

The homomorphism $E_n : C(Q) \to UT_n(Q)$ defined by $E_n(q) = qI_n$ turns $UT_n(Q)$ into a $C(Q)$-algebra, where I_n is the $n \times n$ identity matrix. This allows us to prove the following:

Proposition 2.8. *Let Q be a ring, $f(x), g(x) \in C(Q)[x]$ and let $n \in \mathbb{N}$. Q is $f(x), g(x)$-clean if and only if the ring of upper triangular matrices $UT_n(Q)$ is $f(x), g(x)$-clean.*

Proof. Assume that Q is $f(x), g(x)$-clean and let $W = (w_{ij}) \in UT_n(Q)$. For $i = 1, ..., n$, we can write $w_{ii} = v_{ii} + \eta_{ii} + \theta_{ii}$ where $v_{ii} \in U(Q)$ and $\eta_{ii}, \theta_{ii} \in Q$ such that $f(\eta_{ii}) = g(\theta_{ii}) = 0$. So

$$W = \begin{bmatrix} v_{11} & w_{12} & \dots & w_{1n} \\ 0 & v_{22} & \dots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_{nn} \end{bmatrix} + \begin{bmatrix} \eta_{11} & 0 & \dots & 0 \\ 0 & \eta_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \eta_{nn} \end{bmatrix} + \begin{bmatrix} \theta_{11} & 0 & \dots & 0 \\ 0 & \theta_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \theta_{nn} \end{bmatrix}.$$

We note that the first matrix on the right-hand side is an invertible matrix, the second is a root of $f(x)$ and the third is a root of $g(x)$. Conversely, define $L_n : UT_n(Q) \to Q$ by $L_n(W) = w_{11}$. Obviously, this is a ring surjective morphism. For $q \in Q$, let $D = \text{diag}(q, ..., q)$. Then $L_n(D) = q$ and it is evident now that the $f(x), g(x)$-clean property of q follows from that of D. $\square$

Theorem 2.9. *Let Q be a commutative ring. The following statements hold:*
(i)If Q is a local ring and $f(x), g(x) \in Q[x]$ such that $Z(f) \cap U(Q) = Z(g) \cap U(Q) = \emptyset$, then Q is never $f(x), g(x)$-clean. In particular, Q is clean but not $f(x), g(x)$-clean.
(ii) If $2 \in N(Q)$, then Q is never $(x^2 - r^2), (x^2 + s^2)$-clean for all $r, s \in Q$. In particular, the ring $Q_n = \prod_{i=1}^{n} \mathbb{Z}_{2^i}$ is never $(x^2 - r^2), (x^2 + s^2)$-clean for all $r, s \in Q_n$ and $n \geq 1$.

Proof. (i) On the contrary, suppose that Q is $f(x), g(x)$-clean. So we can write $0 = v + \eta + \theta$ where $v \in U(Q)$ and $\eta, \theta \in Q$ such that $f(\eta) = g(\theta) = 0$. Therefore, $-v = \eta + \theta$. The element $\eta + \theta$ is a unit. So, by the properties of local rings, either η or θ is a unit which is a contradiction.
(ii) Let r, s be arbitrary elements in Q. On the contrary, suppose that Q is $(x^2 - r^2), (x^2 + s^2)$-clean, then we can write $r - s = v + \eta + \theta$ where $v \in U(Q)$ and $\eta, \theta \in Q$ such that $\eta^2 = r^2$ and $\theta^2 = -s^2$. Therefore, $0 = v + (\eta - r) + (\theta + s)$. We note that $(\eta - r)^2 = \eta^2 - 2\eta r + r^2 = r^2 - 2\eta r + r^2 = 2(r^2 - \eta r)$. Similarly, $(\theta + s)^2 = 2\theta s$. From $2 \in N(Q)$, it follows that both $\eta - r$ and $\theta + s$ are nilpotents. Thus $0 = u$, where u is a unit; that is, $0 = 1$ and $Q = 0$ which is a contradiction. In Q_n, we clearly have $2^n = 0$. $\square$

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