International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 653–660

r-Ideals of Commutative Ordered Semigroups

Panuwat Luangchaisri, Thawhat Changphas

Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen 40002, Thailand

email: panulu@kku.ac.th, thacha@kku.ac.th

(Received November 2, 2023, Accepted December 13, 2023,
Published February 12, 2024)

Abstract

In this paper, we investigate r-ideals in commutative ordered semigroups with zero and identity. Moreover, we study some properties of r-ideals. Furthermore, we give several characterizations of r-ideals.

1 Introduction

Mohamadian [4] introduced the concept of r-ideals in commutative rings with $1 \neq 0$, investigated the behavior of r-ideals, and compared them with other classical ideals such as prime and maximal ideals. In [5], the authors considered r-ideals in commutative semigroups with zero 0 and identity 1 such that $1 \neq 0$. Some properties of r-ideals were studied and various characterizations of r-ideals were given. In this paper, r-ideals in commutative ordered semigroups with $1 \neq 0$ are investigated; some properties of r-ideals are studied, and several characterizations of r-ideals are given.

An *ordered semigroup* $(S, ., \leq)$ is a semigroup $(S, .)$ together with a partial order $\leq$ that is compatible with the semigroup operation, meaning that, for any $x, y, z \in S$, if $x \leq y$, then $zx \leq zy$ and $xz \leq yz$. An element 0 in S is called a *zero* (of S) if for any $x \in S$, $0x = x0 = 0$ and $0 \leq x$, and an element 1 in S is called an *identity* (of S) if for any $x \in S$, $1x = x1 = x$.

Key words and phrases: commutative ordered semigroup, ideal, prime ideal, annihilator, r-ideal.
AMS (MOS) Subject Classifications: 06F05, 20M20.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

An ordered semigroup $(S, ., \leq)$ is said to be *commutative* if for any $x, y \in S$, $xy = yx$.

Hereafter, we let $(S, ., \leq)$ *be a commutative ordered semigroups with zero* 0 *and identity* 1 *such that* $1 \neq 0$.

For non-empty subsets A, B of S, define

$$AB = \{xy \mid \ x \in A \text{ and } y \in B\}, \quad (A] = \{y \in S \mid y \leq x \text{ for some } x \in A\}.$$

Observe that $A \subseteq (A]$, $((A]] = (A]$, and $(A](B] \subseteq (AB]$. A non-empty subset I of S is called an *ideal* (of S) if $IS \subseteq I$ (i.e., $xy \in I$, for all $x \in I$ and $y \in S$); and $I = (I]$ (i.e., for any $x \in I$ and $y \in S$ if $y \leq x$, then $y \in I$). An ideal I of S is said to be *proper* if $I \neq S$. Observe that, for any $x \in S$, $(xS]$ is an ideal of S. Suppose that T is a non-empty subset of S. Then

$$(I : T) = \{s \in S \mid st \in I \text{ for every } t \in T\}$$

is an ideal of S. In particular, we write $(I : a)$ instead of $(I : \{a\})$. Furthermore, we write $(0 : a)$ by $ann(a)$. Any element a of S is a *zero divisor* if $ann(a) = \{s \in S \mid sa = 0\} \neq 0$, otherwise a is a *regular element*. The set of all zero divisors in S will be written by $zd(S)$, and the set of all regular elements in S will be written by $r(S)$. The *localization* of S at $r(S)$, written by $Q(S)$, is defined by:

$$Q(S) = \left\{ \tfrac{x}{s} \mid x \in S, s \in r(S) \right\}$$

where $\frac{x}{s} = \frac{x'}{s'} \Leftrightarrow us'x = usx'$ for some $u \in r(S)$. The natural homomorphism is the mapping $\pi : S \to Q(S)$ defined by $\pi(x) = \frac{x}{1}$ for every $x \in S$. If J is an ideal in $Q(S)$, then $J^c = \pi^{-1}(J)$ is an ideal of S, and if I is an ideal of S then the set $I^e = \{\frac{a}{s} \mid a \in I, s \in r(S)\}$ is an ideal of $Q(S)$.

2 Results

We begin this section with the definition of r-ideals of a commutative ordered semigroup with zero 0 and identity 1 such that $0 \neq 1$ as follows:

Definition 2.1. *Let* $(S, ., \leq)$ *be a commutative ordered semigroup with zero* 0 *and identity* 1 *such that* $0 \neq 1$. *A proper ideal* I *of* S *is called an* r-ideal *(of* S*) if for any* $a, b \in S$, $ab \in I$ *and* $ann(a) = 0$ *imply* $b \in I$.

Observe that the zero ideal of S is the k-ideal. Consider the multiplicative ordered semigroup $(\mathbb{Z}_6, ., =)$ of integers modulo 6. Let I be a nonzero proper ideal of $\mathbb{Z}_6$. Let $\bar{a}, \bar{b} \in \mathbb{Z}_6$ be such that $\bar{a}\bar{b} \in I$ and $ann(\bar{a}) = \bar{0}$. Thus, a and 6 are relatively prime and so $\bar{a}$ has an inverse in $\mathbb{Z}_6$, so that $\bar{b} = (\bar{a})^{-1}(\bar{a}\bar{b}) \in I$. Then, every ideal of $\mathbb{Z}_6$ is an r-ideal.

A proper ideal P of S is said to be *prime* if for any $a, b \in S$, $ab \in P$ implies $a \in P$ or $b \in P$. Observe that the notion of prime ideals and r-ideals are different. Indeed, the zero ideal is an r-ideal but not necessarily prime. For example, the zero ideal $\{0\}$ is not a prime ideal in the multiplicative ordered semigroup $(\mathbb{Z}_6, ., =)$, because $\bar{2}\bar{3} = \bar{0} \in \{\bar{0}\}$ but $\bar{2} \notin \{\bar{0}\}$ and $\bar{3} \notin \{\bar{0}\}$. Also, every prime ideal need not be an r-ideal. To illustrate this, consider the multiplicative ordered semigroup $(\mathbb{Z}, ., =)$ and the ideal $I = 3\mathbb{Z}$. Then, I is prime but not an r-ideal since $(3)(1) = 3 \in I$ with $ann(3) = 0$ but $1 \notin I$.

Proposition 2.2. *If I is an r-ideal of S, then $I \subseteq zd(S)$.*

Proof. Assume I is an r-ideal of S and $I \nsubseteq zd(S)$. From $I \nsubseteq zd(S)$, there exists an element $a \in I$ and $ann(a) = 0$. Since $1a = a \in I$ and I is an r-ideal, we have $1 \in I$; so $I = S$. It is a contradiction. Hence $I \subseteq zd(S)$. □

Proposition 2.3. *Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a non-empty set of r-ideals of S. Then the union and the intersection of $\{I_\alpha \mid \alpha \in \Lambda\}$ are r-ideals of S.*

Proof. Let $a, b \in S$ be such that $ab \in \cup_{\alpha\in\Lambda} I_\alpha$ and $ann(a) = 0$. Then, $ab \in I_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Since I_{α_0} is an r-ideal, we have $b \in I_{\alpha_0} \subseteq \cup_{\alpha\in\Lambda} I_\alpha$. For $\cap_{\alpha\in\Lambda} I_\alpha$ is an r-ideal can be proved similarly. □

Observe that a proper ideal P of S is prime if and only if $P = (P : a)$ for every $a \notin P$.

Theorem 2.4. *Let I be a proper ideal of S. The following are equivalent:*

1. *I is an r-ideal.*

2. *$I = (I : r)$ for each $r \in r(S)$.*

3. *$I = J^c$ for some ideal J of $Q(S)$.*

Proof. (1)⇒(2): Suppose I is an r-ideal. Let $r \in r(S)$. Clearly, $I \subseteq (I : r)$. Let $x \in (I : r)$; then $rx \in I$. By assumption, we have $x \in I$. Consequently, $I = (I : r)$.

(2)⇒(1): Let $a, b \in S$ be such that $ab \in I$ and $ann(a) = 0$. By (2), $b \in (I : a) = I$. Therefore, I is an r-ideal.

(1)⇒(3): It is easy to see that $I \subseteq I^{ec}$ for any ideal I of S. Assume I is an r-ideal and $a \in I^{ec}$. Then $\frac{a}{1} \in I^e$; so that $\frac{a}{1} = \frac{x}{s}$ for some $x \in I$ and

$s \in r(S)$. Thus there exists an element $r \in r(S)$ such that $(rs)a = rx \in I$. Since $ann(rs) = 0$, we have $a \in I$. Consequently, $I = I^{ec}$.

(3)⇒(1): Suppose $I = J^c$ where J is an ideal of $Q(S)$. Let $a, b \in S$ be such that $ab \in I$ and $ann(a) = 0$. It is easy to see that $\frac{1}{a} \in Q(S)$ and $\frac{ab}{1} \in J$. Thus, $\frac{1}{a}(\frac{ab}{1}) = \frac{b}{1} \in J$, and so $b \in I = J^c$. □

We call S an *r-po-semigroup* if for any $a, b, c \in S$, $0 < ab \leq ac$ implies that $b \leq uc$ for some unit $u \in S$. An ideal I of S is said to be *pure* if for $x \in I$ there exists $y \in I$ such that $x \leq yx$, see [2]. An ideal I of S is said to be *regular* if for $x \in I$ there exists $y \in I$ such that $x \leq xyx$, see [3].

Proposition 2.5. *Every pure ideals and regular ideals of an r-po-semigroup S are r-ideals.*

Proof. Let I be a pure ideal of an r-po-semigroup S. Let $a, b \in S$ be such that $ab \in I$ and $ann(a) = 0$. If $ab = 0$, then $b = 0 \in I$. Assume $ab \neq 0$. Since I is pure, we have $ab \leq (ab)c$ for some $c \in I$. Then $b \leq u(bc)$ for some unit $u \in S$, because S is an r-po-semigroup. Since $b \leq u(bc) \in I$, we have $b \in I$. Similarly, we have that every regular ideals of an r-po-semigroup S are r-ideals. □

Theorem 2.6. *Let I be a proper ideal of S. Then I is an r-ideal if and only if for ideals J, K of S, $J \cap r(S) \neq \emptyset$ and $JK \subseteq I$ imply $K \subseteq I$.*

Proof. Assume for ideals J, K of S, if $J \cap r(S) \neq \emptyset$ and $JK \subseteq I$, then $K \subseteq I$. Let $a, b \in S$ be such that $ab \in I$ and $ann(a) = 0$. Setting $J = (aS]$ and $K = (bS]$. Then $J \cap r(S) \neq \emptyset$ and

$$JK = (aS](bS] \subseteq (aSbS] = (abSS] \subseteq (abS] \subseteq (I] = I.$$

By assumption, $K \subseteq I$ and so $b \in I$. Conversely, assume I is an r-ideal and let J, K be ideals of S such that $JK \subseteq I$ and $J \cap r(S) \neq \emptyset$. Since $J \cap r(S) \neq \emptyset$, there exists an element $a \in J$ such that $ann(a) = 0$. Note that $aK \subseteq JK \subseteq I$. Since I is an r-ideal and Theorem 2.4, we conclude that $K \subseteq (I : a) = I$. □

Proposition 2.7. *Let I, J be ideals of S such that $I \subseteq J$. If I is an r-ideal of S and J/I is an r-ideal of S/I, then J is an r-ideal of S.*

Proof. Assume I is an r-ideal of S and J/I is an r-ideal of S/I. Let $a, b \in S$ be such that $ab \in J$ and $ann(a) = 0$. We have $(aI)(bI) = (ab)I \in J/I$. To prove that $ann(aI) = 0_{S/I}$, let $r \in S$ be such that $(rI)(aI) = 0_{S/I}$. Then

$(ra)I = 0_{S/I}$. This implies $ra \in I$. Since I is an r-ideal of S, we have $r \in I$, so that $ann(aI) = 0_{S/I}$. Since J/I is an r-ideal of S/I, we get $bI \in J/I$. Hence, $b \in J$. Consequently, J is an r-ideal. □

Theorem 2.8. *Let I be an ideal of S. Then I is an r-ideal if and only if $(I : T)$ is an r-ideal for any $\emptyset \neq T \nsubseteq I$.*

Proof. Assume I is an r-ideal. Let $\emptyset \neq T \nsubseteq I$; then $(I : T) \neq S$. Let $a, b \in S$ be such that $ab \in (I : T)$ and $ann(a) = 0$. Then, $abT \subseteq I$. By assumption, $bT \subseteq I$. So that $b \in (I : T)$, and hence $(I : T)$ is an r-ideal. Conversely, suppose I is not an r-ideal. Then there exist elements $a, b \in S$ such that $ab \in I$ with $ann(a) = 0$, but $b \notin I$. Setting $T = \{b\}$; then $T \nsubseteq I$. Note that $(I : T) \neq S$. Since $ann(a) = 0$, we have $a \notin zd(S)$. Since $ab \in I$, we have $a \in (I : T)$. By Proposition 2.2, $(I : T)$ is not an r-ideal. □

Proposition 2.9. *Let I be an ideal of S with $I \cap r(S) \neq \emptyset$. Then $I \cap J = I \cap K$, where J and K are r-ideals of S, implies that $J = K$.*

Proof. Suppose J, K are r-ideals and I is an ideal of S such that $I \cap J = I \cap K$ with $I \cap r(S) \neq \emptyset$. Since $IJ \subseteq I \cap J = I \cap K \subseteq K$, K is an r-ideal, and Theorem 2.6, we have $J \subseteq K$. Similarly, we have $K \subseteq J$, and so $K = J$. □

We call S an *uz-po-semigroup* if all elements in S is either unit or zero divisor.

Proposition 2.10. *The following are equivalent:*

1. *S is an uz-po-semigroup.*
2. *Every proper principal ideal is an r-ideal.*
3. *Every proper ideal is an r-ideal.*
4. *Every prime ideal is an r-ideal.*
5. *Every maximal ideal is an r-ideal.*

Proof. (1)⇒(2): Suppose I is a proper principal ideal of S. Let $ab \in I$ with $ann(a) = 0$. By (1), a is unit, and so $b = a^{-1}(ab) \in I$.

(2)⇒(3): Assume I is a proper ideal of S and $ab \in I$ with $ann(a) = 0$. Note that $\langle ab \rangle \neq S$. Since $ab \in \langle ab \rangle$ and $\langle ab \rangle$ is an r-ideal of S, we conclude that $b \in \langle ab \rangle \subseteq I$. Hence I is an r-ideal.

(3)⇒(4)⇒(5): It is clear.

(5)⇒(1): Suppose every maximal ideal is an r-ideal. Let a be a nonunit element of S. Then there exists a maximal ideal M containing a. It follows by Proposition 2.2 that $a \in M \subseteq zd(S)$. Hence S is an uz-po-semigroup. □

Suppose S_1, S_2 are ordered semigroups with zero and identity. For $a_1, b_1 \in S_1, a_2, b_2 \in S_2$, define the multiplication on $S_1 \times S_2$ by $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$, and the partial order $\leq$ on $S_1 \times S_2$ by $(a_1, a_2) \leq (b_1, b_2) \Leftrightarrow a_1 \leq b_1$ and $a_2 \leq b_2$. Then $S_1 \times S_2$ becomes an ordered semigroup with zero and identity.

Lemma 2.11. *Let S_1, S_2 be ordered semigroups with zero and identity and $S = S_1 \times S_2$. Suppose that $I = I_1 \times I_2$, where I_1 is an ideal of S_1 and I_2 is an ideal of S_2. Then I is an r-ideal of S if and only if I_1 is an r-ideal of S_1 and $I_2 = S_2$ or $I_1 = S_1$ and I_2 is an r-ideal of S_2 or I_1, I_2 are r-ideals of S_1, S_2, respectively.*

Proof. Assume I is an r-ideal of S. Since I is a proper ideal, we have that at least one of I_1 and I_2 is proper. Without loss of generality we may assume $I_1 = S_1$ and $I_2 \neq S_2$. We show that I_2 is an r-ideal of S_2. The other case can be proved similarly. Let $a_2, b_2 \in S$ be such that $a_2b_2 \in I_2$ and $ann(a_2) = 0$. We have $ann(1, a_2) = 0$ and $(1, a_2)(0, b_2) = (0, a_2b_2) \in I$. Since I is an r-ideal of S, we have $(0, b_2) \in I$, and then $b_2 \in I_2$. Hence I_2 is an r-ideal. Conversely, assume $I = S_1 \times I_2$, where I_2 is an r-ideal of S_2. To prove that I is an r-ideal, let $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2) \in I$ with $ann(a_1, a_2) = 0$. Then $a_2b_2 \in I_2$ and $ann(a_2) = 0$. Since I_2 is an r-ideal of S_2, we conclude that $b_2 \in I_2$. Thus $(b_1, b_2) \in I$, and this completes the proof. Under the other assumptions, one can show that I is an r-ideal. □

Theorem 2.12. *Let $S_1, \ldots, S_n, n \geq 2$ be commutative semigroups with zero and identity and I_i is an ideal of S_i for $1 \leq i \leq n$. Then $I = I_1 \times \ldots \times I_n$ is an r-ideal of $S = S_1 \times S_2 \times \ldots \times S_n$ if and only if I_i is an r-ideal of S_i for some $i \in \{i_1, i_2, \ldots, i_t\}$ and $I_j = S_j$ for every $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_t\}$.*

Proof. We proceed by induction on n. If $n = 2$, then the assertion follows by Lemma 2.11. Suppose the assertion is true for $k \leq n - 1$. Let $k = n$ and $I = I_1 \times \ldots \times I_n$. Setting $J = I_1 \times \ldots \times I_{n-1}$, and $S' = S_1 \times \ldots \times S_{n-1}$. By Lemma 2.11, $I = J \times I_n$ is an r-ideal of $S' \times S_n$ if and only if J is an r-ideal of S' and $I_n = S_n$ or $J = S'$ and I_n is an r-ideal of S_n or J, I_n are r-ideals of S' and S_n, respectively. By induction hypothesis the claim follows. □

Proposition 2.13. *A prime ideal P is an r-ideal if and only if $P \subseteq zd(S)$.*

Proof. If P is a prime ideal of S and is an r-ideal, then by Proposition 2.2 we have $P \subseteq zd(S)$. Assume $P \subseteq zd(S)$. Let $a, b \in S$ be such that $ab \in P$ and $ann(a) = 0$. Since $a \notin P$, we have $b \in P$. □

Proposition 2.14. *Let $P_1, P_2, \ldots, P_n$ be prime ideals of S, which is not comparable, meaning that, $P_i \nsubseteq P_j$ for all $1 \leq i \neq j \leq n$. If $\cap_{i=1}^n P_i$ is an r-ideal, then for any $1 \leq i \leq n$, P_i is an r-ideal.*

Proof. Assume $\cap_{i=1}^n P_i$ is an r-ideal. Without loss of generality we may assume $i = 1$. Let $a, b \in S$ be such that $ab \in P_1$ and $ann(a) = 0$. Since $\cap_{i=2}^n P_i \nsubseteq P_1$, there exists an element $r \in \cap_{i=2}^n P_i$ such that $r \notin P_1$, so $abr \in \cap_{i=1}^n P_i$. Since $\cap_{i=1}^n P_i$ is an r-ideal, we obtain $br \in \cap_{i=1}^n P_i \subseteq P_1$. Thus $b \in P_1$. □

Theorem 2.15. *Let P be an ideal of S. If P is a maximal r-ideal of S, then P is a prime ideal.*

Proof. Assume P is a maximal r-ideal of S. Let $a, b \in S$ be such that $ab \in P$ and $a \notin P$. By Theorem 2.8, we have $(P : a)$ is an r-ideal containing P. By the maximality of P, we have $b \in (P : a) = P$. Hence P is prime. □

We consider the polynomial ordered semigroup $S[x] = \{sx^i \mid s \in S, i \geq 0\}$ defined by an indeterminate x. For any $s, t \in S$ and $i, j \geq 0$, the multiplication on $S[x]$ is defined by $(sx^i)(tx^j) = (st)x^{i+j}$, and the partial order $\leq$ on $S[x]$ is defined by $sx^i \leq tx^j \Leftrightarrow s \leq t$ and $i \leq j$. Observe that if I is an ideal of S, then $I[x] = \{ax^i \mid a \in I, i \geq 0\}$ is an ideal of $S[x]$.

Proposition 2.16. *Let I be a proper ideal of S. Then I is an r-ideal of S if and only if $I[x]$ is an r-ideal of $S[x]$.*

Proof. Suppose I is an r-ideal of S. Let $(sx^i)(tx^j) = (st)x^{i+j} \in I[x]$ with $ann(sx^i) = 0_{S[x]}$. It is easy to see that $ann(sx^i) = 0_{S[x]}$ if and only if $ann(s) = 0$. So, we have $st \in I$ with $ann(s) = 0$. Since I is an r-ideal, we conclude that $t \in I$, and so $tx^j \in I[x]$. The opposite direction is clear. □

The polynomial semigroup $S[x_1, x_2, \ldots, x_n]$ in n variables can be defined in a similar way. Also, one can easily show the following result:

Corollary 2.17. *Let I be a proper ideal of S. Then I is an r-ideal of S if and only if $I[x_1, x_2, \ldots, x_n]$ is an r-ideal of $S[x_1, x_2, \ldots, x_n]$.*

Acknowledgment. The Research on "r-Ideals of commutative ordered semigroups" by Khon Kaen University has received funding support from the National Science, Research and Innovation Fund (NSRF).

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