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$t-$Co-cobalancing Numbers

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Abstract

A positive integer n is called a $t-$co-cobalancing number if n is a solution of the equation $1+2+3+\cdots+(n+1) = (n+1+t)+(n+2+t)+\cdots+(n+r+t)$ for some positive integer r and fixed positive integer t. In this paper, we present a function and recurrence relations for $t-$co-cobalancing numbers. Moreover, we give some interesting properties of $t-$co-cobalancing numbers.

1 Introduction

In 1999, Behera and Panda [1] defined a balancing number as follows:
A positive integer n is called balancing number if

$$1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r) \tag{1.1}$$

for some positive integer r which is called the balancer corresponding to the balancing number n.

In 2005, Panda and Ray [2] defined a cobalancing number $n \in \mathbb{Z}^+$ by

$$1+2+\cdots+n = (n+1)+(n+2)+\cdots+(n+r) \tag{1.2}$$

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for some positive integer r which is called cobalancer corresponding to the balancing number n.

Later in 2012, Dash and Ota [3] studied $t-$balancing numbers. A positive integer n is called a $t-$balancing number if

$$1+2+\cdots+n=(n+1+t)+(n+2+t)+\cdots+(n+r+t) \tag{1.3}$$

for some positive integer r which is called the $t-$balancer.

In 2021, Pakapongpun and Chattae [4] modified (1.1) and (1.2) slightly and called $n \in \mathbb{Z}^+$ a co-cobalancing number if

$$1+2+\cdots+(n+1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1.4}$$

for some positive integer r which is called the co-cobalancer.

The purpose of this paper is to present the notion of the $t-$co-cobalancing number, a function and recurrence relation for them and to give some of their interesting properties.

2 Preliminary Notes

Definition 2.1. *Let d be a positive integer that is not a perfect square. The Pell equation is a Diophantine equation of the form $x^2 - dy^2 = 1$ (More details in [5]).*

Theorem 2.2. *[8] Let (x_1, y_1) be the least positive solution of the Diophantine equation $x^2 - dy^2 = 1$, where d is a positive integer that is not a square. Then all positive solutions (x_k, y_k) are given by*

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k$$

for $k = 1, 2, 3, \cdots$

Theorem 2.3. *If d is a positive integer that is not a perfect square, then equation $x^2 - dy^2 = 1$ has infinitely many solutions in positive integers, and the general solution is given by (x_n, y_n) and $n \geq 0$,*

$$x_{n+1} = x_1x_n + dy_1y_n \text{ and } y_{n+1} = y_1x_n + x_1y_n,$$

where (x_1, y_1) is its fundamental solution; i.e., the minimal solution different from $(1, 0)$ (More details in [6]).

Theorem 2.4. *[7](Brahmagupta's Lemma)*
If (x_1, y_1) is a solution of $dx^2+m_1 = y^2$ and (x_2, y_2) is a solution of $dx^2+m_2 = y^2$, then $(x_1y_2+x_2y_1, y_1y_2+dx_1x_2)$ and $(x_1y_2-x_2y_1, dx_1x_2-y_1y_2)$ are solutions of $dx^2 + m_1m_2 = y^2$.

3 Main Results

A positive integer n is called a $t-$co-cobalancing number if

$$1 + 2 + \cdots + (n + 1) = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t) \quad (3.5)$$

for some positive integer r which is called the $t-$co-cobalancer. The following are some examples of $t-$co-cobalancing number for different values of t:
$5, 34, 203, 1188, 6929$ are 0-co-cobalancing numbers with 0-co-cobalancers $3, 5, 85, 493, 2871$ respectively. $1, 13, 83, 491, 2869$ are 1-co-cobalancing numbers with 1-co-cobalancers $1, 6, 35, 204, 1189$, respectively.
$4, 33, 202, 1187, 6828$ are 2-co-cobalancing numbers with 2-co-cobalancers $2, 14, 8, 492, 2870$, respectively.
$2, 7, 24, 53, 152$ are 3-co-cobalancing numbers with 3-co-cobalancers $1, 3, 10, 22, 63$, respectively.
$5, 10, 44, 73, 271$ are 4-co-cobalancing numbers with 4-co-cobalancers $2, 4, 18, 30, 112$, respectively.
$8, 13, 64, 93, 390$ are 5-co-cobalancing numbers with 5-co-cobalancers $3, 5, 26, 38, 16$, respectively.
From the equation (3.5), we get

$$n = \frac{1}{2}\big[(2r - 3) + \sqrt{8r^2 + 8rt - 8r + 1}\big]. \quad (3.6)$$

Thus r is a $t-$co-cobalancer number if and only if $8r^2 + 8rt - 8r + 1$ is a perfect square.
Let $y = \sqrt{8r^2 + 8rt - 8r + 1}$. Then $y^2 = 8r^2 + 8rt - 8r + 1$. Arranging this, we get $2(2r + t - 1)^2 - y^2 = 2t^2 - 4t + 1$.
Put $x = 2r + t - 1$. Hence

$$2x^2 - y^2 = 2t^2 - 4t + 1. \quad (3.7)$$

The least positive integer solution of (3.7) is $x_1 = t - 1$ and $y_1 = 1$. To find the other solutions of (3.7), consider the Pell equation

$$y^2 - 2x^2 = 1 \quad (3.8)$$

whose fundamental solution is $\overline{x}_1 = 2$ and $\overline{y}_1 = 3$. The other solutions of (3.8) can be derived from the relations $\overline{x}_n = \frac{g_n}{2\sqrt{2}}$ and $\overline{y}_n = \frac{f_n}{2}$, where

$$f_n = (3+2\sqrt{2})^n + (3-2\sqrt{2})^n \text{ and } g_n = (3+2\sqrt{2})^n - (3-2\sqrt{2})^n.$$

Hence

$$\begin{aligned} \overline{x}_n &= \frac{1}{2\sqrt{2}}\left[(3+2\sqrt{2})^n - (3-2\sqrt{2})^n\right] \\ \overline{y}_n &= \frac{1}{2}\left[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n\right], n = 1,2,3\cdots. \end{aligned} \tag{3.9}$$

This shows that both x_n and y_n are positive. Therefore, the following two sets of expressions for x_n and y_n satisfy Brahmagupta's lemma between (x_1, y_1) and $(\overline{x}_n, \overline{y}_n.)$ The other solutions of (3.7) can be obtained from the relations:

$$\begin{aligned} x_n &= x_1\overline{y}_n + y_1\overline{x}_n = (t-1)\overline{y}_n + \overline{x}_n, \quad y_n = y_1\overline{y}_n + dx_1\overline{x}_n = \overline{y}_n + 2(t-1)\overline{x}_n, \\ x'_n &= x_1\overline{y}_n - y_1\overline{x}_n = (t-1)\overline{y}_n - \overline{x}_n, \quad y'_n = dx_1\overline{x}_n - y_1\overline{y}_n = 2(t-1)\overline{x}_n - \overline{y}_n. \end{aligned}$$

Substituting $\overline{x}_n, \overline{y}_n$ of (3.9) into the above equations, we get

$$\begin{aligned} 2\sqrt{2}x_n &= (3+2\sqrt{2})^n(\sqrt{2}(t-1)+1) + (3-2\sqrt{2})^n(\sqrt{2}(t-1)-1), \\ 2\sqrt{2}y_n &= (3+2\sqrt{2})^n(2(t-1)+\sqrt{2}) - (3-2\sqrt{2})^n(2(t-1)-\sqrt{2}), \\ 2\sqrt{2}x'_n &= (3+2\sqrt{2})^n(\sqrt{2}(t-1)-1) + (3-2\sqrt{2})^n(\sqrt{2}(t-1)+1), \\ 2\sqrt{2}y'_n &= (3+2\sqrt{2})^n(2(t-1)-\sqrt{2}) - (3-2\sqrt{2})^n(2(t-1)+\sqrt{2}). \end{aligned}$$

Now, we have obtained two sequences of x_n, y_n and x'_n, y'_n which we can use to get the recurrence relations of x_n, y_n and x'_n, y'_n,

$$\begin{aligned} x_n &= 6x_{n-1} - x_{n-2}, \quad y_n = 6y_{n-1} - y_{n-2}, \\ x'_n &= 6x_{n-1} - x_{n-2}, \quad y'_n = 6y_{n-1} - y_{n-2}. \end{aligned}$$

We denote the n^{th} $t-$co-cobalancing number by $\overline{B}^t_n$. From (3.6) and x_n, y_n, x'_n, y'_n we have

$$\overline{B}^t_n = \frac{1}{2}\left[(2r-3) + \sqrt{8r^2+8rt-8r+1}\right]$$

and

$$y = y_n = y'_n = \sqrt{8r^2+8rt-8r+1}, x = x_n = x'_n = 2r+t-1.$$

Hence

$$\begin{aligned} \overline{B}^t_n &= \frac{1}{2}\left[(2r-3) + x_n + y_n - 2r - t + 1\right] \\ &= \frac{1}{2}\left[x_n + y_n - (t+2)\right], t \geq 3 \end{aligned}$$

which is the generalized recurrence relation of $t-$co-cobalancing numbers.

3.1 Properties of $t-$co-cobalancing numbers

We reduce equation (3.5) to

$$r^2 + r(2n + 2t + 1) - (n^2 + 3n + 2) = 0.$$

n is a $t-$co-cobalancing number if and only if $8n^2 + 8n(t + 2) + (2t + 1)^2 + 8$ is a perfect square. Consider the function

$$F(x) = 3x + (t + 2) + \sqrt{8x^2 + 8x(t + 2) + (2t + 1)^2 + 8}.$$

We will show that for any $t-$co-cobalancing number x, $F(x)$ is a $t-$co-cobalancing number.

Theorem 3.1. *If x is a $t-$co-cobalancing number, then*

$$F(x) = 3x + (t + 2) + \sqrt{8x^2 + 8x(t + 2) + (2t + 1)^2 + 8}$$

is also a $t-$co-cobalancing number.

Proof. Let $F(x) = u$. Thus $x < u$ and

$$x = 3u + (t + 2) - \sqrt{8u^2 + 8u(t + 2) + (2t + 1)^2 + 8}.$$

Since x is a $t-$co-cobalancing number, $8u^2+8u(t+2)+(2t+1)^2+8$ is a perfect square. This implies that u is a $t-$co-cobalancing number and $F(x) = u$. Therefore, $F(x)$ is a $t-$co-cobalancing number. □

Theorem 3.2. *Let $\overline{B}^t_n$ be the n^{th} $t-$co-cobalancing number. If $x = \overline{B}^t_n$, then*

$$\overline{B}^t_{n+2} = F(x) = 3x + (t + 2) + \sqrt{8x^2 + 8x(t + 2) + (2t + 1)^2 + 8}$$

and

$$\overline{B}^t_{n-2} = \overline{F}(x) = 3x + (t + 2) - \sqrt{8x^2 + 8x(t + 2) + (2t + 1)^2 + 8}.$$

Proof. Define a function $F : [-1, \infty) \to [3t - 2, \infty)$ by

$$F(x) = 3x + (t + 2) + \sqrt{8x^2 + 8x(t + 2) + (2t + 1)^2 + 8}.$$

It is clear that $x < F(x)$. Since

$$F'(x) = 3 + \frac{4(2x + t + 2)}{\sqrt{8x^2 + 8x(t + 2) + (2t + 1)^2 + 8}} > 0,$$

F is a strictly increasing function. Hence F is one to one and $x < F(x), x \geq -1$. Thus F^{-1} exists and is also strictly increasing with $F^{-1}(x) < x$. Since

$$F^{-1}(x) = 3x + (t+2) - \sqrt{8x^2 + 8x(t+2) + (2t+1)^2 + 8},$$

$$\begin{aligned} &8(F^{-1}(x))^2 + 8(F^{-1}(x))(t+2) + (2t+1)^2 + 8 \\ &\qquad = \left[3\sqrt{8x^2 + 8x(t+2) + (2t+1)^2 + 8} - 8x - 4(t+2)\right]^2. \end{aligned}$$

It follows that $F^{-1}(x)$ is also a $t-$co-cobalancing number. Next, we will prove the remaining part by mathematical induction.

The first three $t-$co-cobalancing numbers of one of the sequences are $c_1 = 3t - 7, c_2 = 20t - 36, c_3 = 119t - 205$ which generate the odd termed $t-$co-cobalancing numbers and the first three $t-$co-cobalancing numbers of the other sequences are $c_1 = 3t-2, c_2 = 20t-7, c_3 = 119t-2$ which generate the even termed $t-$co-cobalancing numbers. We know that $F(c_1) = c_2$ and $F(c_2) = c_3$. Assume that H_k is the hypothesis that there is no even (or odd) $t-$co-cobalancing number between x_{n-1} and x_n for $n = 1, 2, \cdots, k$. We will prove that H_{k+1} is true; i.e., there is no $t-$co-cobalancing number y such that $x_k < y < x_{k+1}$. Assume to the contrary that there exists a $t-$co-cobalancing number y between $x_k < y < x_{k+1}$. It follows that $F^{-1}(x_k) < F^{-1}(y) < F^{-1}(x_{k+1})$ but $F^{-1}(x_k) = x_{k-1}$ and $F^{-1}(x_{k+1}) = x_k$ thus $x_{k-1} < F^{-1}(y) < x_k$. Since y and $F^{-1}(y)$ are $t-$co-cobalancing numbers, there exists a $t-$co-cobalancing number between x_{k-1} and x_k which is a contradiction. So H_{k+1} is true. Therefore, there is no $t-$co-cobalancing number between x_{k-1} and x_k. □

From the theorem 3.2, if $\overline{B}_n^{\,t} = x$ is an even (or odd) term $t-$co-cobalancing number, then the next even (or odd) term $t-$co-cobalancing number is

$$\overline{B}^t_{n+2} = 3\overline{B}^t_n + (t+2) + \sqrt{8(\overline{B}^t_n)^2 + 8\overline{B}^t_n(t+2) + (2t+1)^2 + 8}$$

and the previous even (or odd) term $t-$co-cobalancing number is

$$\overline{B}^t_{n-2} = 3\overline{B}^t_n + (t+2) - \sqrt{8(\overline{B}^t_n)^2 + 8\overline{B}^t_n(t+2) + (2t+1)^2 + 8}.$$

Then

$$\overline{B}^t_{n+2} = 6\overline{B}^t_n - \overline{B}^t_{n-2} + 2(t+2), t \geq 3.$$

Hence

$$\overline{B}^t_n = 6\overline{B}^t_{n-2} - \overline{B}^t_{n-4} + 2(t+2). \tag{3.10}$$

Theorem 3.3. *If $\overline{B}^t_n$ is the n^{th} $t-$co-cobalancing number, then*

$$\left[\overline{B}^t_n - (t+2)\right]^2 - \overline{B}^t_{n+2} \cdot \overline{B}^t_{n-2} = (2t+1)^2 + 8.$$

Proof. From equation (3.10), we have

$$\overline{B}^t_{n+2} = 6\overline{B}^t_n - \overline{B}^t_{n-2} + 2(t+2).$$

Thus

$$\frac{\overline{B}^t_{n+2} + \overline{B}^t_{n-2} - 2(t+2)}{\overline{B}^t_n} = 6 \tag{3.11}$$

and

$$\frac{\overline{B}^t_n + \overline{B}^t_{n-4} - 2(t+2)}{\overline{B}^t_{n-2}} = 6. \tag{3.12}$$

From equations (3.11) and (3.12), we get

$$\begin{aligned}&(\overline{B}^t_n)^2 + \overline{B}^t_n \cdot \overline{B}^t_{n-4} - 2\overline{B}^t_n(t+2)\\ &\quad = (\overline{B}^t_{n-2})^2 + \overline{B}^t_{n+2} \cdot \overline{B}^t_{n-2} - 2\overline{B}^t_{n-2}(t+2).\end{aligned}$$

Hence

$$\left[\overline{B}^t_n - (t+2)\right]^2 - \overline{B}^t_{n+2} \cdot \overline{B}^t_{n-2} = \left[\overline{B}^t_{n-2} - (t+2)\right]^2 - \overline{B}^t_n \cdot \overline{B}^t_{n-4}.$$

Similarly,

$$\left[\overline{B}^t_{n-2} - (t+2)\right]^2 - \overline{B}^t_n \cdot \overline{B}^t_{n-4} = \left[\overline{B}^t_{n-4} - (t+2)\right]^2 - \overline{B}^t_{n-2} \cdot \overline{B}^t_{n-6}.$$

Continuing in the same way, we get

$$[\overline{B}^t_n - (t+2)]^2 - \overline{B}^t_{n+2} \cdot \overline{B}^t_{n-2} = \begin{cases} [\overline{B}^t_3 - (t+2)]^2 - \overline{B}^t_5 \cdot \overline{B}^t_1 \text{ if } n \text{ is odd} \\ [\overline{B}^t_4 - (t+2)]^2 - \overline{B}^t_6 \cdot \overline{B}^t_2 \text{ if } n \text{ is even.} \end{cases}$$

We know that $\overline{B}^t_1 = 3t-7, \overline{B}^t_3 = 20t-36, \overline{B}^t_5 = 119t-205$ and $\overline{B}^t_2 = 3t-2, \overline{B}^t_4 = 20t-7, \overline{B}^t_6 = 119t-2$.
Substituting these values in both cases, we have

$$\left[\overline{B}^t_n - (t+2)\right]^2 - \overline{B}^t_{n+2} \cdot \overline{B}^t_{n-2} = (2t+1)^2 + 8.$$

□

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