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On (τ_1, τ_2)-R_1 bitopological spaces

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Abstract

In this paper, we deal with the notion of (τ_1, τ_2)-R_1 bitopological spaces. Moreover, some characterizations of (τ_1, τ_2)-R_1 bitopological spaces are investigated.

1 Introduction

Davis [7] introduced the notion of a separation axiom called R_1. Shanin [18] studied the notion of R_0 topological spaces. These notions are further investigated by Naimpally [16], Dube [11] and Dorsett [8]. Murdeshwar and Naimpally [15] and Dube [10] studied some properties of the class of R_1 topological spaces. As natural generalizations of the separations axioms R_0 and R_1, the notions of semi-R_0 and semi-R_1 spaces were introduced and studied

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by Maheshwari and Prasad [14] and Dorsett [9]. Caldas et al. [6] introduced and studied two new weak separation axioms called Λ_θ-R_0 and Λ_θ-R_1 by using the notions of (Λ, θ)-open sets and the (Λ, θ)-closure operator. Cammaroto and Noiri [5] defined a weak separation axiom m-R_0 in m-spaces which are equivalent to generalized topological spaces due to Lugojan [13]. Noiri [17] introduced the notion of m-R_1 spaces and investigated several characterizations of m-R_0 spaces and m-R_1 spaces. Thongmoon and Boonpok [20] introduced and studied the notion of (Λ, p)-R_1 topological spaces. Furthermore, some characterizations of sober $\delta p(\Lambda, s)$-R_0 spaces were investigated in [19]. In [1], the authors introduced and studied the notions of $\delta s(\Lambda, s)$-R_0 spaces and $\delta s(\Lambda, s)$-R_1 spaces. Moreover, several characterizations of Λ_p-R_0 spaces and (Λ, s)-R_0 spaces were presented in [3] and [2], respectively. In this paper, we introduce the notion of (τ_1, τ_2)-R_1 bitopological spaces. In particular, some characterizations of (τ_1, τ_2)-R_1 bitopological spaces are discussed.

2 Preliminaries

Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [4] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [4] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [4] of A and is denoted by $\tau_1\tau_2$-Int(A). The set $\cap\{G \mid A \subseteq G$ and G is $\tau_1\tau_2$-open$\}$ is called the $\tau_1\tau_2$-*kernel* [4] of A and is denoted by $\tau_1\tau_2$-$ker(A)$.

Lemma 2.1. [4] *For subsets A, B of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-$ker(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2$-$ker(A) \subseteq \tau_1\tau_2$-$ker(B)$.

(3) If A is $\tau_1\tau_2$-open, then $\tau_1\tau_2$-$ker(A) = A$.

(4) $x \in \tau_1\tau_2$-$ker(A)$ if and only if $A \cap H \neq \emptyset$ for every $\tau_1\tau_2$-closed set H containing x.

3 Characterizations of (τ_1, τ_2)-R_1 spaces

In this section, we introduce the notion of (τ_1, τ_2)-R_1 spaces. Moreover, some characterizations of (τ_1, τ_2)-R_1 spaces are discussed.

Definition 3.1. *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-R_1 if for each points x and y in X with $\tau_1\tau_2$-$Cl(\{x\}) \neq \tau_1\tau_2$-$Cl(\{y\})$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $\tau_1\tau_2$-$Cl(\{x\}) \subseteq U$ and $\tau_1\tau_2$-$Cl(\{y\}) \subseteq V$.*

Definition 3.2. [12] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-R_0 if for each $\tau_1\tau_2$-open set U and each $x \in U$, $\tau_1\tau_2$-$Cl(\{x\}) \subseteq U$.*

Lemma 3.3. *If a bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_1, then it is (τ_1, τ_2)-R_0.*

Proof. The proof follows from Theorem 5.1 of [17]. □

Lemma 3.4. [12] *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_0 if and only if for each points x and y in X, $\tau_1\tau_2$-$Cl(\{x\}) \neq \tau_1\tau_2$-$Cl(\{y\})$ implies*

$$\tau_1\tau_2\text{-}Cl(\{x\}) \cap \tau_1\tau_2\text{-}Cl(\{y\}) = \emptyset.$$

Theorem 3.5. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_1 if and only if for each points x and y in X with $\tau_1\tau_2$-$Cl(\{x\}) \neq \tau_1\tau_2$-$Cl(\{y\})$, there exist $\tau_1\tau_2$-closed sets F_x and F_y such that $x \in F_x$, $y \notin F_x$, $y \in F_y$, $x \notin F_y$ and $X = F_x \cup F_y$.*

Proof. Let x and y be any points in X with $\tau_1\tau_2$-Cl$(\{x\}) \neq \tau_1\tau_2$-Cl$(\{y\})$. There exist disjoint $\tau_1\tau_2$-open sets U_x and U_y such that $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U_x$ and $\tau_1\tau_2$-Cl$(\{y\}) \subseteq U_y$. Now, put $F_x = X - U_y$ and $F_y = X - U_x$. Then F_x and F_y are $\tau_1\tau_2$-closed sets of X such that $x \in F_x$, $y \notin F_x$, $y \in F_y$, $x \notin F_y$ and $X = F_x \cup F_y$.

Conversely, let x and y be any points in X with $\tau_1\tau_2$-Cl$(\{x\}) \neq \tau_1\tau_2$-Cl$(\{y\})$. Then $\tau_1\tau_2$-Cl$(\{x\}) \cap \tau_1\tau_2$-Cl$(\{y\}) = \emptyset$. In fact, if

$$z \in \tau_1\tau_2\text{-Cl}(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{y\}),$$

then $\tau_1\tau_2$-Cl$(\{z\}) \neq \tau_1\tau_2$-Cl$(\{x\})$ or $\tau_1\tau_2$-Cl$(\{z\}) \neq \tau_1\tau_2$-Cl$(\{y\})$. In case $\tau_1\tau_2$-Cl$(\{z\}) \neq \tau_1\tau_2$-Cl$(\{x\})$, by the hypothesis, there exists a $\tau_1\tau_2$-closed set F such that $x \in F$ and $z \notin F$. Then $z \in \tau_1\tau_2$-Cl$(\{x\}) \subseteq F$. This contradicts that $z \notin F$. In case $\tau_1\tau_2$-Cl$(\{z\}) \neq \tau_1\tau_2$-Cl$(\{y\})$, similarly, this leads to the contradiction. Thus $\tau_1\tau_2$-Cl$(\{x\}) \cap \tau_1\tau_2$-Cl$(\{y\}) = \emptyset$. By Lemma 3.4, (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. By the hypothesis, there exist $\tau_1\tau_2$-closed sets F_x and F_y such that $x \in F_x$, $y \notin F_x$, $y \in F_y$, $x \notin F_y$ and $X = F_x \cup F_y$. Put $U_x = X - F_y$ and $U_y = X - F_x$. Then U_x and U_y are $\tau_1\tau_2$-open sets of X containing x and y, respectively. Since (X, τ_1, τ_2) is (τ_1, τ_2)-R_0, we have $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U_x$, $\tau_1\tau_2$-Cl$(\{y\}) \subseteq U_y$ and also $U_x \cap U_y = \emptyset$. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-R_1. □

Definition 3.6. [12] *Let (X, τ_1, τ_2) be a bitopological space and $x \in X$. Then $\langle x\rangle_{(\tau_1,\tau_2)}$ is defined by $\langle x\rangle_{(\tau_1,\tau_2)} = \tau_1\tau_2$-Cl$(\{x\}) \cap \tau_1\tau_2$-ker$(\{x\})$.*

Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called $(\tau_1, \tau_2)\theta$-*cluster point* [21] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the (τ_1, τ_2)-*closure* [21] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A).

Lemma 3.7. [12] *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_0 if and only if $\langle x\rangle_{(\tau_1,\tau_2)} = \tau_1\tau_2$-Cl$(\{x\})$ for each $x \in X$.*

Theorem 3.8. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_1 if and only if $\langle x\rangle_{(\tau_1,\tau_2)} = (\tau_1, \tau_2)\theta$-Cl$(\{x\})$ for each $x \in X$.*

Proof. Let (X, τ_1, τ_2) be (τ_1, τ_2)-R_1. By Lemma 3.3, (X, τ_1, τ_2) is (τ_1, τ_2)-R_0 and by Lemma 3.7, $\langle x\rangle_{(\tau_1,\tau_2)} = \tau_1\tau_2$-Cl$(\{x\}) \subseteq (\tau_1, \tau_2)\theta$-Cl$(\{x\})$ for each $x \in X$. Thus $\langle x\rangle_{(\tau_1,\tau_2)} \subseteq (\tau_1, \tau_2)\theta$-Cl$(\{x\})$ for each $x \in X$. In order to show the opposite inclusion, suppose that $y \notin \langle x\rangle_{(\tau_1,\tau_2)}$. Then $\langle x\rangle_{(\tau_1,\tau_2)} \neq \langle y\rangle_{(\tau_1,\tau_2)}$. Since (X, τ_1, τ_2) is (τ_1, τ_2)-R_0, by Lemma 3.7, $\tau_1\tau_2$-Cl$(\{x\}) \neq \tau_1\tau_2$-Cl$(\{y\})$. Since (X, τ_1, τ_2) is (τ_1, τ_2)-R_1, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$ and $\tau_1\tau_2$-Cl$(\{y\}) \subseteq V$. Since $\tau_1\tau_2$-Cl$(V) \cap \{x\} \subseteq \tau_1\tau_2$-Cl$(V) \cap U = \emptyset$, $y \notin (\tau_1, \tau_2)\theta$-Cl$(\{x\})$. Thus $(\tau_1, \tau_2)\theta$-Cl$(\{x\}) \subseteq \langle x\rangle_{(\tau_1,\tau_2)}$ and hence $(\tau_1, \tau_2)\theta$-Cl$(\{x\}) = \langle x\rangle_{(\tau_1,\tau_2)}$.

Conversely, suppose that $\langle x\rangle_{(\tau_1,\tau_2)} = (\tau_1, \tau_2)\theta$-Cl$(\{x\})$ for each $x \in X$. Then $\langle x\rangle_{(\tau_1,\tau_2)} = (\tau_1, \tau_2)\theta$-Cl$(\{x\}) \supseteq \tau_1\tau_2$-Cl$(\{x\}) \supseteq \langle x\rangle_{(\tau_1,\tau_2)}$ for each $x \in X$. By Lemma 3.7, (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. Suppose that

$$\tau_1\tau_2\text{-Cl}(\{x\}) \neq \tau_1\tau_2\text{-Cl}(\{y\}).$$

Then by Lemma 3.4, $\tau_1\tau_2$-Cl$(\{x\}) \cap \tau_1\tau_2$-Cl$(\{y\}) = \emptyset$. By Lemma 3.7, $\langle x\rangle_{(\tau_1,\tau_2)} \cap \langle x\rangle_{(\tau_1,\tau_2)} = \emptyset$ and hence $(\tau_1, \tau_2)\theta$-Cl$(\{x\}) \cap (\tau_1, \tau_2)\theta$-Cl$(\{y\}) = \emptyset$. Since $y \notin (\tau_1, \tau_2)\theta$-Cl$(\{x\})$, there exists a $\tau_1\tau_2$-open set U_y such that $y \in U_y \subseteq \tau_1\tau_2$-Cl$(U_y) \subseteq X - \{x\}$. Let $U_x = X - \tau_1\tau_2$-Cl(U_y). Then U_x is $\tau_1\tau_2$-open and $x \in U_x$. Since (X, τ_1, τ_2) is (τ_1, τ_2)-R_0, $\tau_1\tau_2$-Cl$(\{y\}) \subseteq U_y$, $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U_x$ and $U_x \cap U_y = \emptyset$. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-R_1. □

Corollary 3.9. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_1 if and only if $\tau_1\tau_2$-Cl$(\{x\}) = (\tau_1, \tau_2)\theta$-Cl$(\{x\})$ for each $x \in X$.*

Proof. Let (X, τ_1, τ_2) be (τ_1, τ_2)-R_1. By Theorem 3.8, we have

$$\tau_1\tau_2\text{-Cl}(\{x\}) \supseteq \langle x\rangle_{(\tau_1,\tau_2)} = (\tau_1, \tau_2)\theta\text{-Cl}(\{x\}) \supseteq \tau_1\tau_2\text{-Cl}(\{x\})$$

and hence $\tau_1\tau_2$-Cl$(\{x\}) = (\tau_1, \tau_2)\theta$-Cl$(\{x\})$ for each $x \in X$.

Conversely, suppose that $\tau_1\tau_2$-Cl$(\{x\}) = (\tau_1, \tau_2)\theta$-Cl$(\{x\})$ for each $x \in X$. First, we show that (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. Let U be a $\tau_1\tau_2$-open set and $x \in U$. Let $y \notin U$. Then $\tau_1\tau_2$-Cl$(\{y\}) \cap U = (\tau_1, \tau_2)\theta$-Cl$(\{y\}) \cap U = \emptyset$. Thus $x \notin (\tau_1, \tau_2)\theta$-Cl$(\{y\})$. There exists a $\tau_1\tau_2$-open set V such that $x \in V$ and $y \notin \tau_1\tau_2$-Cl(V). Since $\tau_1\tau_2$-Cl$(\{x\}) \subseteq \tau_1\tau_2$-Cl$(V)$, $y \notin \tau_1\tau_2$-Cl$(\{x\})$. This shows that $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$ and (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. By Lemma 3.7, $\langle x\rangle_{(\tau_1,\tau_2)} = \tau_1\tau_2$-Cl$(\{x\}) = (\tau_1, \tau_2)\theta$-Cl$(\{x\})$ for each $x \in X$ and by Theorem 3.8, (X, τ_1, τ_2) is (τ_1, τ_2)-R_1. □

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