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Properties of $(\tau_1, \tau_2)\star$-closed sets

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Abstract

In this paper, we introduce the notion of $(\tau_1, \tau_2)\star$-closed sets. Moreover, we investigate some properties of $(\tau_1, \tau_2)\star$-closed sets and $(\tau_1, \tau_2)\star$-open sets.

1 Introduction

The notion of generalized closed sets was first introduced by Levine [11]. A subset A of a topological space X is called generalized closed if $\mathrm{Cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is open. Moreover, Levine [11] studied some properties of generalized closed sets and generalized open sets. In [13], the present

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authors introduced and investigated the notions of generalized (Λ, p)-closed sets and generalized (Λ, p)-open sets. Some properties of generalized (Λ, α)-closed sets, generalized $\delta p(\Lambda, s)$-closed sets, generalized (Λ, s)-closed sets and generalized (Λ, sp)-closed sets were studied in [1], [2], [3] and [4], respectively. Kelly [10] introduced the notion of bitopological spaces. Such spaces are equipped with two topologies. Generalized closed sets and generalized open sets are extended to bitopological spaces by Fukutake [7]. Dungthaisong et al. [6] introduced and studied the notions of $\mu_{(m,n)}$-closed sets and $\mu_{(m,n)}$-open sets in bigeneralized topological spaces. Jafari and Rajesh [8] introduced and investigated the notion of generalized closed sets with respect to an ideal in ideal topological spaces. A subset A of an ideal topological space X is called generalized closed with respect to an ideal if $\mathrm{Cl}(A) - U \in \mathscr{I}$, whenever $A \subseteq U$ and U is open. Noiri and Rajesh [12] introduced and studied the notion of generalized closed sets with respect to an ideal in ideal bitopological spaces. In this paper, we introduce the notion of $(\tau_1, \tau_2)\star$-closed sets. Moreover, we discuss some properties of $(\tau_1, \tau_2)\star$-closed sets and $(\tau_1, \tau_2)\star$-open sets.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-$\mathrm{Cl}(A)$ and τ_i-$\mathrm{Int}(A)$, respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [5] if $A = \tau_1$-$\mathrm{Cl}(\tau_2$-$\mathrm{Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [5] of A and is denoted by $\tau_1\tau_2$-$\mathrm{Cl}(A)$. The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [5] of A and is denoted by $\tau_1\tau_2$-$\mathrm{Int}(A)$.

Lemma 2.1. [5] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-$Cl(A)$ and $\tau_1\tau_2$-$Cl(\tau_1\tau_2$-$Cl(A)) = \tau_1\tau_2$-$Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2$-$Cl(A) \subseteq \tau_1\tau_2$-$Cl(B)$.

(3) $\tau_1\tau_2$-$Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-$Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A nonempty collection $\mathscr{I}$ of subsets of X is called *an ideal* [9] if satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ implies $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$.

3 Properties of $(\tau_1, \tau_2)\star$-closed sets

In this section, we introduce the notion of $(\tau_1, \tau_2)\star$-closed sets. Moreover, some properties of $(\tau_1, \tau_2)\star$-closed sets and $(\tau_1, \tau_2)\star$-open sets are discussed.

Definition 3.1. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. A subset A of X is said to be $(\tau_1, \tau_2)\star$-closed if $\tau_1\tau_2\text{-}Cl(A) - U \in \mathscr{I}$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open.*

Theorem 3.2. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. A subset A of X is $(\tau_1, \tau_2)\star$-closed if and only if $F \subseteq \tau_1\tau_2\text{-}Cl(A) - A$ and F is $\tau_1\tau_2$-closed in X implies $F \in \mathscr{I}$.*

Proof. Let F be a $\tau_1\tau_2$-closed set and $F \subseteq \tau_1\tau_2\text{-Cl}(A) - A$. Then $A \subseteq X - F$. By hypothesis, $\tau_1\tau_2\text{-Cl}(A) - (X - F) \in \mathscr{I}$. Since $F \subseteq \tau_1\tau_2\text{-Cl}(A) - (X - F)$, we have $F \in \mathscr{I}$.

Conversely, suppose that $F \subseteq \tau_1\tau_2\text{-Cl}(A) - A$ and F is $\tau_1\tau_2$-closed in X implies $F \in \mathscr{I}$. Let U be a $\tau_1\tau_2$-open set and $A \subseteq U$. Then

$$\tau_1\tau_2\text{-Cl}(A) - U = \tau_1\tau_2\text{-Cl}(A) \cap (X - U)$$

is a $\tau_1\tau-2$-closed in X, that is contained in $\tau_1\tau_2\text{-Cl}(A) - A$. By the hypothesis, $\tau_1\tau_2\text{-Cl}(A) - U \in \mathscr{I}$. Thus A is $(\tau_1, \tau_2)\star$-closed. □

Theorem 3.3. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. If A and B are $(\tau_1, \tau_2)\star$-closed in X, then $A \cup B$ is $(\tau_1, \tau_2)\star$-closed.*

Proof. Suppose that A and B are $(\tau_1, \tau_2)\star$-closed. Let U be a $\tau_1\tau_2$-open set and $A \cup B \subseteq U$. Then we have $A \subseteq U$ and $B \subseteq U$. By the hypothesis, $\tau_1\tau_2\text{-Cl}(A) - U \in \mathscr{I}$ and $\tau_1\tau_2\text{-Cl}(B) - U \in \mathscr{I}$. Thus

$$\tau_1\tau_2\text{-Cl}(A \cup B) - U = [\tau_1\tau_2\text{-Cl}(A) - U] \cup [\tau_1\tau_2\text{-Cl}(B) - U] \in \mathscr{I}.$$

This shows that $A \cup B$ is $(\tau_1, \tau_2)\star$-closed. □

Theorem 3.4. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. If A is a $(\tau_1, \tau_2)\star$-closed set and F is a $\tau_1\tau_2$-closed set of X, then $A \cap F$ is $(\tau_1, \tau_2)\star$-closed.*

Proof. Let V be a $\tau_1\tau_2$-open set and $A \cup F \subseteq V$. Then $A \subseteq V \cup (X - F)$. Since A is $(\tau_1, \tau_2)\star$-closed, we have $\tau_1\tau_2\text{-Cl}(A) - (V \cup (X - F)) \in \mathscr{I}$. Now, $\tau_1\tau_2\text{-Cl}(A \cap F) \subseteq \tau_1\tau_2\text{-Cl}(A) \cap F = (\tau_1\tau_2\text{-Cl}(A) \cap F) - (X - F)$. Thus

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(A \cap F) - V &\subseteq \tau_1\tau_2\text{-Cl}(A) \cap F - (V \cap (X - F)) \\ &\subseteq \tau_1\tau_2\text{-Cl}(A) - (V \cup (X - F)) \in \mathscr{I}\end{aligned}$$

and hence $A \cap F$ is $(\tau_1, \tau_2)\star$-closed. □

Theorem 3.5. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. If A is $(\tau_1, \tau_2)\star$-closed in X and $A \subseteq B \subseteq \tau_1\tau_2$-Cl$(A)$, then B is $(\tau_1, \tau_2)\star$-closed.*

Proof. Let V be a $\tau_1\tau_2$-open set and $B \subseteq V$. Then, $A \subseteq V$. Since A is $(\tau_1, \tau_2)\star$-closed, we have $\tau_1\tau_2\text{-Cl}(A) - V \in \mathscr{I}$. Now $B \subseteq \tau_1\tau_2\text{-Cl}(A)$ implies that $\tau_1\tau_2\text{-Cl}(B) - V \subseteq \tau_1\tau_2\text{-Cl}(A) - V \in \mathscr{I}$. Thus B is $(\tau_1, \tau_2)\star$-closed. □

Definition 3.6. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. A subset A of X is said to be $(\tau_1, \tau_2)\star$-open if $X - A$ is $(\tau_1, \tau_2)\star$-closed.*

Theorem 3.7. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. A subset A of X is $(\tau_1, \tau_2)\star$-open if and only if $F - V \subseteq \tau_1\tau_2$-Int$(A)$ for some $V \in \mathscr{I}$, whenever $F \subseteq A$ and F is $\tau_1\tau_2$-closed.*

Proof. Let F be a $\tau_1\tau_2$-closed set and $F \subseteq A$. Then we have $X - A \subseteq X - F$. By the hypothesis, $\tau_1\tau_2\text{-Cl}(X - A) \subseteq (X - F) \cup V$ for some $V \in \mathscr{I}$. Thus $X - ((X - F) \cup V) \subseteq X - \tau_1\tau_2\text{-Cl}(X - A)$ and hence $F - V \subseteq \tau_1\tau_2\text{-Int}(A)$.

Conversely, let G be a $\tau_1\tau_2$-open set and $X - A \subseteq G$. Then $X - G \subseteq A$. By the hypothesis, $(X - G) - V \subseteq \tau_1\tau_2\text{-Int}(A) = X - \tau_1\tau_2\text{-Cl}(X - A)$ for some $V \in \mathscr{I}$. This gives that $X - (G \cup V) \subseteq X - \tau_1\tau_2\text{-Cl}(X - A)$. Therefore, $\tau_1\tau_2\text{-Cl}(X - A) \subseteq G \cup V$ for some $V \in \mathscr{I}$. Thus $\tau_1\tau_2\text{-Cl}(X - A) - G \in \mathscr{I}$. This shows that $X - A$ is $(\tau_1, \tau_2)\star$-closed and hence A is $(\tau_1, \tau_2)\star$-open. □

Theorem 3.8. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. If A and B are $(\tau_1, \tau_2)\star$-open sets of X such that $\tau_1\tau_2$-Cl$(A) \cap B = \emptyset$ and $\tau_1\tau_2$-Cl$(B) \cap A = \emptyset$, then $A \cup B$ is $(\tau_1, \tau_2)\star$-open.*

Proof. Let F be a $\tau_1\tau_2$-closed set and $F \subseteq A \cup B$. Then $\tau_1\tau_2\text{-Cl}(A) \cap F \subseteq A$ and $\tau_1\tau_2\text{-Cl}(B) \cap F \subseteq B$. By the hypothesis, $\tau_1\tau_2\text{-Cl}(A) \cap F - U \subseteq \tau_1\tau_2\text{-Int}(A)$ and $\tau_1\tau_2\text{-Cl}(B) \cap F - V \subseteq \tau_1\tau_2\text{-Int}(B)$ for some $U, V \in \mathscr{I}$. This means that $\tau_1\tau_2\text{-Cl}(A) \cap F - \tau_1\tau_2\text{-Int}(A) \in \mathscr{I}$ and $\tau_1\tau_2\text{-Cl}(B) \cap F - \tau_1\tau_2\text{-Int}(B) \in \mathscr{I}$. Thus $[(\tau_1\tau_2\text{-Cl}(A) \cap F - \tau_1\tau_2\text{-Int}(A)) \cup (\tau_1\tau_2\text{-Cl}(B) \cap F - \tau_1\tau_2\text{-Int}(B))] \in \mathscr{I}$ and hence $[F \cap (\tau_1\tau_2\text{-Cl}(A) \cup \tau_1\tau_2\text{-Cl}(B)) - (\tau_1\tau_2\text{-Int}(A) \cup \tau_1\tau_2\text{-Int}(B))] \in \mathscr{I}$. Since $F = (A \cup B) \cap F \subseteq \tau_1\tau_2\text{-Cl}(A \cup B) \cap F$, we have

$$\begin{aligned} F - \tau_1\tau_2\text{-Int}(A \cup B) &\subseteq [\tau_1\tau_2\text{-Cl}(A \cup B) \cap F] - \tau_1\tau_2\text{-Int}(A \cup B) \\ &\subseteq [(\tau_1\tau_2\text{-Cl}(A \cup B) \cap F) - (\tau_1\tau_2\text{-Int}(A) \cup \tau_1\tau_2\text{-Int}(B))] \in \mathscr{I} \end{aligned}$$

and hence $F - G \subseteq \tau_1\tau_2\text{-Int}(A \cup B)$ for some $G \in \mathscr{I}$. This proves that $A \cup B$ is $(\tau_1, \tau_2)\star$-open. □

Theorem 3.9. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. If A is $(\tau_1, \tau_2)\star$-open in X and $\tau_1\tau_2$-Int$(A) \subseteq B \subseteq A$, then B is $(\tau_1, \tau_2)\star$-open.*

Proof. Suppose that A is $(\tau_1, \tau_2)\star$-open and $\tau_1\tau_2\text{-Int}(A) \subseteq B \subseteq A$. Then $X - A \subseteq X - B \subseteq \tau_1\tau_2\text{-Cl}(X - A)$ and $X - A$ is $(\tau_1, \tau_2)\star$-closed. By Theorem 3.5, $X - B$ is $(\tau_1, \tau_2)\star$-closed and hence B is $(\tau_1, \tau_2)\star$-open. □

Theorem 3.10. *Let (X, τ_1, τ_2) be a bitopological space and $\mathscr{I}$ be an ideal on X. A subset A of X is $(\tau_1, \tau_2)\star$-closed if and only if $\tau_1\tau_2$-Cl$(A) - A$ is $(\tau_1, \tau_2)\star$-open.*

Proof. Suppose that $F \subseteq \tau_1\tau_2\text{-Cl}(A) - A$ and F is $\tau_1\tau_2$-closed. Then we have $F \in \mathscr{I}$. This implies that $F - V = \emptyset$ for some $V \in \mathscr{I}$. Thus $F - V \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A) - A)$. By Theorem 3.7, $\tau_1\tau_2\text{-Cl}(A) - A$ is $(\tau_1, \tau_2)\star$-open.

Conversely, let V be a $\tau_1\tau_2$-open set and $A \subseteq V$. Then

$$\tau_1\tau_2\text{-Cl}(A) \cap (X - V) \subseteq \tau_1\tau_2\text{-Cl}(A) \cap (X - A) = \tau_1\tau_2\text{-Cl}(A) - A.$$

By the hypothesis, $[\tau_1\tau_2\text{-Cl}(A) \cap (X - V)] - G \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A) - A) = \emptyset$ for some $G \in \mathscr{I}$. Thus $\tau_1\tau_2\text{-Cl}(A) \cap (X - V) \subseteq G \in \mathscr{I}$ and hence $\tau_1\tau_2\text{-Cl}(A) - G \in \mathscr{I}$. This shows that A is $(\tau_1, \tau_2)\star$-closed. □

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