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(τ_1, τ_2)-extremal disconnectedness in bitopological spaces

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Abstract

In this paper, we introduce the notion of (τ_1, τ_2)-extremally disconnected spaces. Moreover, we establish some characterizations of (τ_1, τ_2)-extremally disconnected spaces.

1 Introduction

The notion of extremally disconnected spaces was introduced by Stone [9]. A topological space X is called extremally disconnected if the closure of every open set of X is open or equivalently if the interior of every closed set of X is closed. Extremally disconnected spaces play a prominent role in set-theoretical topology, in the theory of Boolean algebras, and in some

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branches of functional analysis. Sivaraj [8] obtained some characterizations of extremally disconnected spaces by utilizing smi-open sets. Noiri [7] investigated several characterizations of extremally disconnected spaces by utilizing preopen sets and semi-preopen sets. Ekici and Noiri [5] introduced and studied the notion of $\star$-extremally disconnected ideal topological spaces. Furthermore, Ekici and Noiri [5] showed that $\star$-extremally disconnectedness and extremally disconnectedness are equivalent for a codense ideal. Viriyapong and Boonpok [10] introduced and studied the notion of (Λ, p)-extremal disconnectedness in topological spaces. Moreover, Kong-ied and Boonpok [6] investigated some characterizations of (Λ, p)-extremally disconnected spaces. Several characterizations of (Λ, s)-extremally disconnected spaces and Λ_{sp}-extremally disconnected spaces were established in [1] and [2], respectively. In this paper, we introduce the notion of (τ_1, τ_2)-extremally disconnected spaces. In particular, we investigate some characterizations of (τ_1, τ_2)-extremally disconnected spaces.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [4] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [4] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [4] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [4] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)*s-open* [3] (resp. (τ_1, τ_2)*p-open* [3], $(\tau_1, \tau_2)\beta$*-open* [3]) if $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). A subset A of a bitopological space (X, τ_1, τ_2) is called (τ_1, τ_2)*r-open* [11] (resp. (τ_1, τ_2)*r-closed*) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A = \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$*-open* if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$.

3 On (τ_1, τ_2)-extremally disconnected spaces

In this section, we introduce the notion of (τ_1, τ_2)-extremally disconnected spaces. Moreover, we discuss several characterizations of (τ_1, τ_2)-extremally disconnected spaces.

Definition 3.1. *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-extremally disconnected if the $\tau_1\tau_2$-closure of every $\tau_1\tau_2$-open set U of X is $\tau_1\tau_2$-open.*

Theorem 3.2. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected.

(2) $\tau_1\tau_2$-Int(F) is $\tau_1\tau_2$-closed for every $\tau_1\tau_2$-closed set F of X.

(3) $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ for every subset A of X.

(4) Every (τ_1, τ_2)s-open set of X is (τ_1, τ_2)p-open.

(5) The $\tau_1\tau_2$-closure of every $(\tau_1, \tau_2)\beta$-open set of X is $\tau_1\tau_2$-open.

(6) Every $(\tau_1, \tau_2)\beta$-open set of X is (τ_1, τ_2)p-open.

(7) For every subset A of X, A is $\alpha(\tau_1, \tau_2)$-open if and only if it is (τ_1, τ_2)s-open.

Proof. (1) $\Rightarrow$ (2): Let F be a $\tau_1\tau_2$-closed set. Then $X - F$ is $\tau_1\tau_2$-open and by (1), $\tau_1\tau_2\text{-Cl}(X - F) = X - \tau_1\tau_2\text{-Int}(F)$ is $\tau_1\tau_2$-open. Thus $\tau_1\tau_2$-Int(F) is $\tau_1\tau_2$-closed.

(2) $\Rightarrow$ (3): Let A be any subset of X. Then $X - \tau_1\tau_2\text{-Int}(A)$ is $\tau_1\tau_2$-closed and by (2), $\tau_1\tau_2\text{-Int}(X - \tau_1\tau_2\text{-Int}(A))$ is $\tau_1\tau_2$-closed. Thus $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$ is $\tau_1\tau_2$-open and hence $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)) \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$.

(3) $\Rightarrow$ (4): Let U be a $(\tau_1, \tau_2)s$-open set. By (3), we have

$$U \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(U)) \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)).$$

Thus U is $(\tau_1, \tau_2)p$-open.

(4) $\Rightarrow$ (5): Let U be a $(\tau_1, \tau_2)\beta$-open set. Then $\tau_1\tau_2$-Cl(U) is $(\tau_1, \tau_2)s$-open and by (4), $\tau_1\tau_2$-Cl(U) is $(\tau_1, \tau_2)p$-open. Thus

$$\tau_1\tau_2\text{-Cl}(U) \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))$$

and hence $\tau_1\tau_2$-Cl(U) is $\tau_1\tau_2$-open.

(5) $\Rightarrow$ (6): Let U be a $(\tau_1, \tau_2)\beta$-open set. Then by (5),

$$\tau_1\tau_2\text{-Cl}(U) = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)).$$

Thus $U \subseteq \tau_1\tau_2$-Cl$(U) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U))$ and hence U is $(\tau_1, \tau_2)p$-open.

(6) $\Rightarrow$ (7): Let U be a $(\tau_1, \tau_2)s$-open set. Since a $(\tau_1, \tau_2)s$-open set is $(\tau_1, \tau_2)\beta$-open, thus by (6) it is $(\tau_1, \tau_2)p$-open. Since U is $(\tau_1, \tau_2)s$-open and $(\tau_1, \tau_2)p$-open, U is $\alpha(\tau_1, \tau_2)$-open.

(7) $\Rightarrow$ (1): Let U be a $\tau_1\tau_2$-open set. Then $\tau_1\tau_2$-Cl(U) is $(\tau_1, \tau_2)s$-open and by (7), $\tau_1\tau_2$-Cl(U) is $\alpha(\tau_1, \tau_2)$-open. Thus

$$\tau_1\tau_2\text{-Cl}(U) \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))$$

and hence $\tau_1\tau_2$-Cl$(U) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U))$. Therefore, $\tau_1\tau_2$-Cl(U) is $\tau_1\tau_2$-open. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected. □

Theorem 3.3. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected.

(2) $\tau_1\tau_2$-Cl$(U) \cap \tau_1\tau_2$-Cl$(V) = \emptyset$ for every $\tau_1\tau_2$-open sets U and V with $U \cap V = \emptyset$;

(3) $\tau_1\tau_2$-Cl$(U) \cap \tau_1\tau_2$-Cl$(V) \subseteq \tau_1\tau_2$-Cl$(U \cap V)$ for every $\tau_1\tau_2$-open sets U and V;

(4) $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))) \cap \tau_1\tau_2$-Cl$(U) = \emptyset$ for every subset $A \subseteq X$ and every $\tau_1\tau_2$-open set U with $A \cap U = \emptyset$.

Proof. (1) $\Rightarrow$ (3): Let U and V be $\tau_1\tau_2$-open. Since $\tau_1\tau_2$-Cl(U) and V are $\tau_1\tau_2$-open,

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(U) \cap \tau_1\tau_2\text{-Cl}(V) &\subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Cl}(U) \cap V)\\ &\subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Cl}(U \cap V)) \subseteq \tau_1\tau_2\text{-Cl}(U \cap V).\end{aligned}$$

Thus $\tau_1\tau_2$-Cl$(U) \cap \tau_1\tau_2$-Cl$(V) \subseteq \tau_1\tau_2$-Cl$(U \cap V)$.

(3) $\Rightarrow$ (2): Let U and V be $\tau_1\tau_2$-open with $U \cap V = \emptyset$. By (3), we have $\tau_1\tau_2$-Cl$(U) \cap \tau_1\tau_2$-Cl$(V) \subseteq \tau_1\tau_2$-Cl$(U \cap V) \subseteq \tau_1\tau_2$-Cl$(\emptyset) = \emptyset$. Thus

$$\tau_1\tau_2\text{-Cl}(U) \cap \tau_1\tau_2\text{-Cl}(V) = \emptyset.$$

(2) $\Rightarrow$ (4): Let $A \subseteq X$ and U be a $\tau_1\tau_2$-open set with $A \cap U = \emptyset$. Since $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ is $\tau_1\tau_2$-open and $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)) \cap U = \emptyset$, by (2), $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))) \cap \tau_1\tau_2$-Cl$(U) = \emptyset$.

(4) $\Rightarrow$ (2): Let U and V be $\tau_1\tau_2$-open with $U \cap V = \emptyset$. By (4), we have $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U))) \cap \tau_1\tau_2$-Cl$(V) = \emptyset$. Since $\tau_1\tau_2$-Cl$(U) \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U)))$, $\tau_1\tau_2$-Cl$(U) \cap \tau_1\tau_2$-Cl$(V) = \emptyset$.

(2) $\Rightarrow$ (1): Let U be a $\tau_1\tau_2$-open set. Since U and $X - \tau_1\tau_2$-Cl(U) are disjoint $\tau_1\tau_2$-open sets. Then $\tau_1\tau_2$-Cl$(U) \cap \tau_1\tau_2$-Cl$(X - \tau_1\tau_2$-Cl$(U)) = \emptyset$. This implies that $\tau_1\tau_2$-Cl$(U) \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U))$. Thus $\tau_1\tau_2$-Cl(U) is $\tau_1\tau_2$-open and hence (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected. □

Theorem 3.4. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected.

(2) Every $(\tau_1, \tau_2)r$-open set of X is $\tau_1\tau_2$-closed.

(3) Every $(\tau_1, \tau_2)r$-closed set of X is $\tau_1\tau_2$-open.

Proof. (1) $\Rightarrow$ (2): Let U be a $(\tau_1, \tau_2)r$-open set of X. Since U is $\tau_1\tau_2$-open, by (1), $\tau_1\tau_2$-Cl(U) is $\tau_1\tau_2$-open. Thus $U = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U)) = \tau_1\tau_2$-Cl(U) and hence U is $\tau_1\tau_2$-closed.

(2) $\Rightarrow$ (1): Let U be a $\tau_1\tau_2$-open set of X. Since $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U))$ is $(\tau_1, \tau_2)r$-open and by (2), $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U))$ is $\tau_1\tau_2$-closed. Thus

$$\tau_1\tau_2\text{-Cl}(U) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))$$

and hence $\tau_1\tau_2$-Cl(U) is $\tau_1\tau_2$-open. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected.

(2) $\Leftrightarrow$ (3): Obvious. □

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