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$\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed sets in bitopological spaces

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Abstract

Our main purpose is to introduce the notion of $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed sets. Moreover, we study some properties of $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed sets.

1 Introduction

The notions of closed sets and open sets are fundamental in the investigation of general topology. Levine [12] introduced the notion of generalized closed sets. This notion has been studied extensively in recent years by many topologists because generalized closed sets are only natural generalizations of closed sets. Dunham and Levine [9] investigated further properties of generalized

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closed sets. Moreover, Levine defined a separation axiom called $T_{\frac{1}{2}}$ between T_0 and T_1. As a modification of generalized closed sets, Palaniappan and Rao [14] introduced and studied the notion of regular generalized closed sets. Dungthaisong et al. [8] investigated the notion of generalized closed sets in bigeneralized topological spaces and studied some characterizations of pairwise μ-$T_{\frac{1}{2}}$ spaces. Viriyapong and Boonpok [15] introduced and investigated the notion of generalized (Λ, p)-closed sets. Furthermore, some properties of generalized (Λ, α)-closed sets, generalized $\delta p(\Lambda, s)$-closed sets, generalized (Λ, s)-closed sets and generalized (Λ, sp)-closed sets were studied in [2], [3], [4] and [5], respectively. Maki [13] called a subset A of a topological space (X, τ) a Λ-set if it is the intersection of open sets containing A. Arenas et al. [1] defined a subset A to be λ-closed if $A = L \cap F$, where L is a Λ-set and F is closed in (X, τ). Ganster et al. [10] introduced and studied the notion of pre-Λ-sets in topological spaces. In this paper, we introduce the notion of $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed sets. Moreover, some properties of $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed sets are investigated.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [6] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [6] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [6] of A and is denoted by $\tau_1\tau_2$-Int(A). The set $\cap\{G \mid A \subseteq G$ and G is $\tau_1\tau_2$-open$\}$ is called the $\tau_1\tau_2$-*kernel* [6] of A and is denoted by $\tau_1\tau_2$-$ker(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is called a $\Lambda_{(\tau_1,\tau_2)}$-*set* [7] if $A = \tau_1\tau_2\text{-}ker(A)$.

3 Properties of $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed sets

In this section, we introduce the notion of $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed sets. Moreover, we discuss some properties of $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed sets.

Definition 3.1. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed if $A = U \cap F$, where U is a $\Lambda_{(\tau_1,\tau_2)}$-set and F is a $\tau_1\tau_2$-closed set of X.*

Lemma 3.2. [7] *For subsets A and $B_\gamma(\gamma \in \Gamma)$ of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $\tau_1\tau_2$-$ker(A)$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.

(2) If A is a $\tau_1\tau_2$-open set, then A is a $\Lambda_{(\tau_1,\tau_2)}$-set.

(3) If B_γ is a $\Lambda_{(\tau_1,\tau_2)}$-set for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} B_\gamma$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.

(4) If B_γ is a $\Lambda_{(\tau_1,\tau_2)}$-set for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma} B_\gamma$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.

Theorem 3.3. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) A is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed;

(2) $A = \tau_1\tau_2$-$Cl(A) \cap U$, where U is a $\Lambda_{(\tau_1,\tau_2)}$-set;

(3) $A = \tau_1\tau_2$-$Cl(A) \cap \tau_1\tau_2$-$ker(A)$.

Proof. (1) ⇒ (2): Let $A = U \cap F$, where U is a $\Lambda_{(\tau_1,\tau_2)}$-set and F is a $\tau_1\tau_2$-closed set of X. Since $A \subseteq F$, we have $\tau_1\tau_2$-Cl$(A) \subseteq F$ and $A = U \cap F \supseteq \tau_1\tau_2$-Cl$(A) \cap U \supseteq A$. Thus $A = \tau_1\tau_2$-Cl$(A) \cap U$.

(2) ⇒ (3): Let $A = \tau_1\tau_2$-Cl$(A) \cap U$, where U is a $\Lambda_{(\tau_1,\tau_2)}$-set. Since $A \subseteq U$, we have $\tau_1\tau_2$-$ker(A) \subseteq \tau_1\tau_2$-$ker(U) = U$ and hence

$$A \subseteq \tau_1\tau_2\text{-}ker(A) \cap \tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(A) \cap U = A.$$

Therefore, $A = \tau_1\tau_2$-Cl$(A) \cap \tau_1\tau_2$-$ker(A)$.

(3) ⇒ (1): By Lemma 3.2, $\tau_1\tau_2$-$ker(A)$ is a $\Lambda_{(\tau_1,\tau_2)}$-set and $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed. By (3), $A = \tau_1\tau_2$-Cl$(A) \cap \tau_1\tau_2$-$ker(A)$ and hence A is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed. □

Lemma 3.4. *Let (X, τ_1, τ_2) be a bitopological space. Then the following properties hold:*

(1) Every $\Lambda_{(\tau_1,\tau_2)}$-set of X is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed.

(2) Every $\tau_1\tau_2$-closed set of X is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed.

Lemma 3.5. *Let (X, τ_1, τ_2) be a bitopological space and $\{A_\gamma \mid \gamma \in \Gamma\}$ be a family of subsets of X. If A_γ is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma} A_\gamma$ is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed.*

Proof. Suppose that A_γ is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed for each $\gamma \in \Gamma$. Then, for each $\gamma \in \Gamma$, there exists a $\Lambda_{(\tau_1,\tau_2)}$-set U_γ and a $\tau_1\tau_2$-closed set F_γ such that $A_\gamma = U_\gamma \cap F_\gamma$. Thus $\cap_{\gamma\in\Gamma} A_\gamma = \cap_{\gamma\in\Gamma}(U_\gamma \cap F_\gamma) = (\cap_{\gamma\in\Gamma} U_\gamma) \cap (\cap_{\gamma\in\Gamma} F_\gamma)$. By Lemma 3.2, $\cap_{\gamma\in\Gamma} U_\gamma$ is a $\Lambda_{(\tau_1,\tau_2)}$-set and $\cap_{\gamma\in\Gamma} F_\gamma$ is $\tau_1\tau_2$-closed. This shows that $\cap_{\gamma\in\Gamma} A_\gamma$ is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed. □

Definition 3.6. [7] *A bitopological space (X, τ_1, τ_2) is said to be:*

(1) (τ_1, τ_2)-T_0 if for any pair of distinct points in X, there exists a $\tau_1\tau_2$-open set containing one of the points but not the other.

(2) (τ_1, τ_2)-T_1 if for any pair of distinct points x, y in X, there exist $\tau_1\tau_2$-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Theorem 3.7. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-T_0 if and only if for each $x \in X$, the singleton $\{x\}$ is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed.*

Proof. Suppose that (X, τ_1, τ_2) is (τ_1, τ_2)-T_0. For each $x \in X$, we have $\{x\} \subseteq \tau_1\tau_2\text{-}ker(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{x\})$. If $y \neq x$, we have (i) there exists a $\tau_1\tau_2$-open set U_x such that $x \in U_x$ and $y \notin U_x$ or (ii) there exists a $\tau_1\tau_2$-open set U_y such that $x \notin U_y$ and $y \notin U_y$. In case (i), $y \notin \tau_1\tau_2\text{-}ker(\{x\})$ and hence $y \notin \tau_1\tau_2\text{-}ker(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{x\})$. This shows that

$$\{x\} \supseteq \tau_1\tau_2\text{-}ker(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{x\}).$$

Thus $\{x\} = \tau_1\tau_2\text{-}ker(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{x\})$ and by Theorem 3.3, $\{x\}$ is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed.

Conversely, suppose that (X, τ_1, τ_2) is not (τ_1, τ_2)-T_0. There exist distinct points x, y such that (i) $y \in U_x$ for every $\tau_1\tau_2$-open set U_x containing x and (ii) $x \in U_y$ for every $\tau_1\tau_2$-open set U_y containing y. By (i) and (ii), $y \in \tau_1\tau_2\text{-}ker(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{x\}) = \{x\}$ by Theorem 3.3. This is contrary to $x \neq y$. □

Definition 3.8. [11] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-R_0 if for each $\tau_1\tau_2$-open set U and each $x \in U$, $\tau_1\tau_2$-$Cl(\{x\}) \subseteq U$.*

Lemma 3.9. [7] *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_1*;*

(2) (X, τ_1, τ_2) *is* (τ_1, τ_2)-R_0 *and* (τ_1, τ_2)-T_0*;*

(3) $(X, \Lambda_{(\tau_1,\tau_2)})$ *is* R_0 *and* T_0*;*

(4) $(X, \Lambda_{(\tau_1,\tau_2)})$ *is* T_1*;*

(5) $(X, \Lambda_{(\tau_1,\tau_2)})$ *is discrete.*

Lemma 3.10. [7] *For a bitopological space* (X, τ_1, τ_2)*, the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_1*;*

(2) for each $x \in X$*, the singleton* $\{x\}$ *is* $\tau_1\tau_2$*-closed in* X*;*

(3) for each $x \in X$*, the singleton* $\{x\}$ *is a* $\Lambda_{(\tau_1,\tau_2)}$*-set.*

Corollary 3.11. *Let* (X, τ_1, τ_2) *be a* (τ_1, τ_2)-R_0 *space. For each* $x \in X$*, the following properties are equivalent:*

(1) The singleton $\{x\}$ *is* $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$*-closed.*

(2) (X, τ_1, τ_2) *be a* (τ_1, τ_2)-T_1*.*

(3) The singleton $\{x\}$ *is* $\tau_1\tau_2$*-closed.*

Proof. (1) $\Rightarrow$ (2): By Theorem 3.7, (X, τ_1, τ_2) is (τ_1, τ_2)-T_0 and (τ_1, τ_2)-R_0 and hence by Lemma 3.9, (X, τ_1, τ_2) is (τ_1, τ_2)-T_1.

(2) $\Rightarrow$ (3): By Lemma 3.10, the singleton $\{x\}$ is $\tau_1\tau_2$-closed.

(3) $\Rightarrow$ (1): By Lemma 3.4, the singleton $\{x\}$ is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed. □

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