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Sequential Total Network Interactions: Connectivity and Proximity Properties

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Abstract

In this paper, we model a network obtained by sequential total linear interactions of independent sequence of finite networks. We determined the connectivity of the nodes and the proximity of every pair of nodes of the resulting network structure.

1 Introduction

Network analysis plays a very important role in optimization problems. Networks considered in this study are simple and undirected. We denote a

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network by $N = \langle n(N), l(N) \rangle$, where $n(N)$ is the set of nodes of N, called the *node-set* of N and $l(N)$ is the set of links of N, called the *link-set* of N. The *order* of N is the number of nodes of N and is denoted by $|n(N)|$. The *size* of N is the number of links of N and is denoted by $|l(N)|$. A *link* in N joins two nodes of N. If u and v are joined in N, then we say that uv is a link in N and we write $uv \in l(N)$. In this case, u and v are *adjacent*. We say that a network is *simple* if it has no loops and no multiple links. A *loop* in a network N is an link in N from a node to itself. If u and v are joined by more than one link, then we say that u and v are connected by multiple links. We say that N is *connected* (with respect to adjacency) if for every $u, v \in n(N)$, there is a path in N joining u and v. A *path* in N is a sequence $[v_1, v_1v_2, v_2, v_2v_3, v_3, v_3v_4, \ldots, v_{n-1}, v_{n-1}v_n, v_n]$ of nodes and links in N. A path network of order n with nodes $v_1, v_2, \ldots, v_n$ (in this order) is simply denoted by $P_n = [v_1, v_2, v_3, \ldots, v_n]$.

A subset S of $n(N)$ is said to be *independent* in N if the elements of S are pairwise non-adjacent in N. This means that for any two nodes $u, v \in S$, $uv \notin l(N)$. In this case, we call S an *independent set*.

A network M is a *subnetwork* of N if $n(M) \subseteq n(N)$ and $l(M) \subseteq l(N)$. A subnetwork M of N is called an *induced subnetwork* of N if any two nodes in M are adjacent in M if and only if they are adjacent in N. If S is a subset of $n(N)$, then a subnetwork with node-set S and with adjacency in S follows from the adjacency in N, then the network obtained in this manner is an induced subnetwork with vertex-set S. In this case, we write the subnetwork induced by S in N as $\langle S \rangle_N$ or simply $\langle S \rangle$ when there is no confusion. We write $\langle S \rangle_N$ to mean that S is a subset of $n(N)$.

Let N be a simple and undirected network. The connectivity of a node u of N, denoted by $con_N(u)$, is the number of nodes of N adjacent to u. A network N is said to be *normalized* if the proximity between two adjacent nodes is 1. This means that the weight/length of the link joining them is 1. The *proximity* between two nodes u and v in a normalized network N, denoted by $prox_N(u, v)$, is the length of a shortest path joining them. If there is no path joining u and v in N, then $prox_N(u, v) = +\infty$. Equivalently, the proximity between u and v is the geodetic distance between u and v as defined in [1].

2 Results

A sequence of finite networks $\langle N_i\rangle_{i=1}^{k}$ is said to be *independent* if $[n(N_s)] \cap [n(N_t)] = \varnothing$ for $s \neq t$. Let $I = \{1, 2, \ldots, k\}$ be an indexing set. The *sequential total interactions* of independent networks $N_1, N_2, \ldots, N_k$ is the network

$$\overset{\leftrightarrow}{\bigoplus_{i\in I}} N_i = N_1 \overset{\leftrightarrow}{\oplus} N_2 \overset{\leftrightarrow}{\oplus} \cdots \overset{\leftrightarrow}{\oplus} N_k$$

with *node-set*

$$n\left(\overset{\leftrightarrow}{\bigoplus_{i\in I}} N_i\right) = \bigcup_{i=1}^{k} n(N_i)$$

and *link-set*

$$l\left(\overset{\leftrightarrow}{\bigoplus_{i\in I}} N_i\right) = \bigcup_{i=1}^{k} l(N_i) \cup \bigcup_{i=1}^{k-1} \{uv : u \in n(N_i), v \in n(N_{i+1})\}.$$

From the above definition, the *order* of $\overset{\leftrightarrow}{\bigoplus_{i\in I}} N_i$ is

$$\left| n\left(\overset{\leftrightarrow}{\bigoplus_{i\in I}} N_i\right)\right| = \sum_{i=1}^{k} |n(N_i)|$$

and the *size* is

$$\left| l\left(\overset{\leftrightarrow}{\bigoplus_{i\in I}} N_i\right)\right| = \sum_{i=1}^{k} |l(N_i)| + \sum_{i=1}^{k-1} |n(N_i)||n(N_{i+1})|.$$

For $k = 2$, the sequential total interaction of N_1 and N_2 is equivalent to the join of N_1 and N_2 (viewed as graphs) as defined in the book by Harary [2].

Based on the above definition of sequential total network interactions of a sequence of finite networks, we have the following results.

Lemma 2.1. *Let $\langle N_i\rangle_{i=1}^{k}$ be a sequence of finite networks and $u \in n(N_s), v \in n(N_t) with s \neq t$. Then $dist_{\overset{\leftrightarrow}{\bigoplus}_{i\in I} N_i}(u, v) = |s - t|$.*

Lemma 2.2. *Let $\langle N_i\rangle_{i=1}^{k} \subseteq \mathcal{N}$ be a sequence of finite networks and $\{u, v\} \subseteq n(\overset{\leftrightarrow}{\bigoplus}_{i\in I} N_i)$. Then $dist_{\overset{\leftrightarrow}{\bigoplus}_{i\in I} N_i}(u, v) = 1$ if and only if one of the following holds:*

(*i*) $uv \in l(N_i)$ *for some* $i \in \{1, 2, \ldots, k\}$.

(*ii*) *There exists* $s \in \{1, 2, \ldots, k-1\}$ *such that* $u \in n(N_s)$ *and* $v \in n(N_{s+1})$.

The following result characterizes the connectivity of every node in $\overset{\leftrightarrow}{\bigoplus}_{i\in I} N_i$.

Theorem 2.3. *Let* $\langle Ni \rangle_{i=1}^{k}$ *be a sequence of finite networks and* $N = \overset{\leftrightarrow}{\bigoplus_{i\in I}} N_i$. *Then the following hold:*

(*i*) *For every* $u \in n(N_1)$, $con_N(u) = con_{N_1}(u) + |n(N_2)|$.

(*ii*) *For every* $v \in n(N_k)$, $con_N(v) = con_{N_k}(v) + |n(N_{k-1})|$.

(*iii*) *For every* $w \in N_i$, *where* $i \in \{2, 3, \ldots, k-1\}$, $con_N(w) = con_{N_i}(w) + |n(N_{i-1})| + |n(N_{i+1})|$.

Proof.

(*i*) Let $u \in n(N_1)$. The result follows from the connectivity of u in N_1 and since u is also connected to all nodes in N_2.

(*ii*) Let $v \in n(N_k)$. The result follows from the connectivity of v in N_k and since v is also connected to all nodes in N_{k-1}.

(*iii*) Let $i \in \{2, 3, \ldots, k-1\}$ and let $w \in n(N_i)$. Then w is connected to all nodes in N_{i-1} and N_{i+1}. Adding its connectivity in N_i, we have the desired result.

The proof is complete. □

The following result establishes the proximity of pairs of nodes in $\overset{\leftrightarrow}{\bigoplus_{i\in I}} N_i$.

Theorem 2.4. *Let* $\langle Ni \rangle_{i=1}^{k}$ *be a sequence of finite networks and* $N = \overset{\leftrightarrow}{\bigoplus_{i\in I}} N_i$. *Then the following hold:*

(*i*) *If* $u \in n(N_s)$ *and* $v \in n(N_t)$ *with* $s \neq t$, *then* $prox_N(u, v) = |s - t|$.

(*ii*) *If* $\{u, v\} \subseteq n(N_i)$ *for some* $i \in \{1, 2, \ldots, k\}$, *then*

(*a*) $prox_N(u, v) = 1$ *whenever* $uv \in l(N_i)$

(*b*) $prox_N(u,v) = 2$ *whenever* $uv \notin l(N_i)$.

Proof. Let $u, v \in n(N)$.

(*i*) Suppose without loss of generality that $u \in n(N_s)$ and $v \in n(N_t)$ where $s < t$. Consider the sequence of networks $\langle N_s, N_{s+1}, N_{s+2}, \ldots, N_{t-1}, N_t \rangle$. For each pair (x, y) with $x \in n(N_p)$ and $y \in n(N_{p+1})$ for $p \in \{s, s+1, \ldots, t-1\}$, $prox_G(x,y) = 1$. Hence, $prox_N(u,v) = t - s = |s - t| = |t - s|$.

(*ii*) (*a*) This is clear since $uv \in l\left(\overset{\leftrightarrow}{\bigoplus_{i \in I}} N_i\right)$ whenver $uv \in l(N_i)$ for some $i \in \{1, 2, \ldots, k\}$.

(*b*) Suppose $uv \notin l(N_i)$. If $i = 1$, then for every $w \in n(N_2)$, $uw, wv \in l(N)$. Hence, $[u, w, v]$ is a u-v geodesic in N. Hence, $prox_N(u,v) = 2$. If $i > 1$, then every $z \in n(N_{i-1})$, $[u, z, v]$ is a u-v geodesic in N. Hence, $prox_N(u,v) = 2$.

The proof is complete. □

References

[1] A.M. Asdain, J.I.C. Salim, R.G. Artes Jr., Geodetic Bounds in Graphs, *International Journal of Mathematics and Computer Science*, **18,** no. 4, (2023), 767–771.

[2] F. Harary, *Graph Theory*, CRC Press, Boca Raton, 2018.

[3] M.E. Van Valkenburg, *Network Analysis,* Prentice Hall, New Jersey, 1974.