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Applications of Chuh-Shih-Chieh's Identity in Geodetic Independence Polynomials

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Abstract

Let G be a simple connected graph. The geodetic independence polynomial G, denoted by $g(G;x)$, is the polynomial whose coefficients correspond to the number of geodetic independent subsets of $V(G)$. In this paper, we apply the Chuh-Shih-Chieh's Identity to establish the geodetic independence polynomial of cycles.

1 Introduction

Graph polynomials garnered a lot of attention in recent years because of some applications of these polynomials in other fields of sciences such as Chemistry, Biology, and Physics [2]. In 1994, Hoede and Li [4] introduced the concept of independent set polynomial of graphs, counting the number of independent subsets of the vertex-set of graphs. In line with the concept of geodetic sets in graphs, Laja and Artes [5] pioneered a study on convex subgraph polynomials. In their work, the concept of geodetic sets had been instrumental in obtaining their results. Recently, Artes et al. [1] introduced the concept of geodetic closure polynomial of graphs.

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In this present work, we consider the independence of geodetics. For two vertices u and v in a graph G, the *geodetic closure* of $\{u, v\}$ is the set $I_G[u, v] = \{u, v\} \cup \{y : y$ lies in a u-v shortest path in $G\}$. A subset S of $V(G)$ is said to be a *geodetic independent set* if $I_G(S) \cap S = \varnothing$. The *geodetic independence number* k of a graph G is the maximum cardinality of the geodetic independent subsets of $V(G)$. The *geodetic independence polynomial (GIP)* of a graph G is given by $g(G; x) = \sum_{i=0}^{k} g_i(G)x^i$, where $g_i(G)$ is the number of geodetic independent subsets of cardinality i.

2 Results

We will use the following combinatorial identity to establish our results.

Lemma 2.1 (Chuh-Shih-Chieh's Identity). *For $n, r \in \mathbb{N}$, $n > r$,*

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}.$$

The following result characterizes the geodetic independent subsets of $V(C_{2k+1})$.

Lemma 2.2. *Let $k \in \mathbb{N}$. A subset S of $V(C_{2k+1})$ is geodetic independent in C_{2k+1} if and only if $|S| \leq 2$ or all vertices do not lie on a path of length less than $k+1$ when $|S| = 3$.*

Proof: All subsets of cardinality less than or equal to 2 are geodetic independent. When S is a 3-element geodetic independent subset, it is clear that one vertex must not lie on the geodetic of the other two vertices. Now, we will show that there are no such sets of cardinality greater than 3. It is enough to show that there are no geodetic independent subsets of cardinality 4. Let $S_3 = \{w_1, w_2, w_3\}$ be a geodetic independent subset of $V(C_{2k+1})$. Note that $diam(C_{2k+1}) = k$. So, $d(w_i, w_j) \leq k$. If a fourth vertex w_4 is included in S_3, then w_4 will lie on one of the paths $[w_1, \ldots, w_2]$, $[w_2, \ldots, w_3]$ and $[w_1, \ldots, w_3]$. Let $S_4 = \{w_1, w_2, w_3, w_4\}$ and, without loss of generality, let w_4 be on $w_1, \ldots, w_2$. This means that $I_{C_{2k+1}}(w_1, w_2) \neq \varnothing$ and $I_{C_{2k+1}}(S_4) \cap S_4 \neq \varnothing$. It follows that there are no geodetic independent subsets of $V(C_{2k+1})$ of cardinality 4.

Now, suppose all vertices lie on a path of length less than $k+1$ when $|S| = 3$. For $x, y, z \in S$, let y be in between x and z. It follows that

$d_{C_{2k+1}}(x,y) + d_{C_{2k+1}}(y,z) \leq k$. Hence, S is not geodetic independent. Thus, the second part of the necessity holds. □

Theorem 2.3. *Let $k \in \mathbb{N}$. Then,*

$$g(C_{2k+1};x) = \sum_{i=0}^{2} \binom{2k+1}{i} x^i + \binom{2k+1}{3}\binom{k+1}{2} x^3.$$

Proof: Let S_3 denote a 3-element geodetic independent subset of $V(C_{2k+1})$ with $S_3 = \{u_1, u_2, u_3\}$. Consider the graph below:

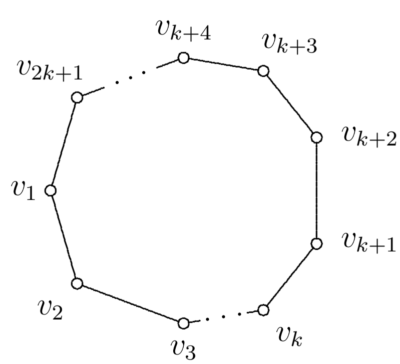

Figure 1: Cycle C_{2k+1} of order $2k+1$

Take $u_1 = v_1$ as a fixed point. The choices for the second vertex lie on the path of length k with v_1 as one of its endpoints. Without loss of generality, let this path be $v_1, v_2, \ldots, v_{k+1}$. If $u_2 = v_2$, count k vertices after v_1 counterclockwise and count k vertices after v_2 clockwise. These vertices form the path $v_{k+3}, v_{k+4}, \ldots, v_{2k+1}, v_1, \ldots, v_{k+1}$. This means that v_{k+2} is the only possible candidate for u_3. By repeating the steps above, we deduce that $u_3 \in \{v_{k+2}, v_{k+3}\}$ whenever $u_2 = v_3$, $u_3 \in \{v_{k+2}, v_{k+3}, v_{k+4}\}$ whenever $u_2 = v_4$, ..., and $u_3 \in \{v_{k+2}, \ldots, v_{2k+1}\}$ whenever $u_2 = v_{k+1}$. This implies that, with fixed vertex $u_1 = v_1$, there are $1+2+\cdots+[(2k+1)-(k+2)+1] = 1+2+\cdots+k = \binom{k+1}{2}$ geodetic independent sets by Chuh-Shih-Chieh's Identity. Thus, there are $\binom{2k+1}{3}\binom{k+1}{2}$ geodetic independent subsets of $V(C_{2k+1})$ of cardinality 3. □

The following result characterizes the geodetic independent subsets of $V(C_{2k})$.

Lemma 2.4. *Let $k \geq 2$. A subset S of $V(C_{2k})$ is geodetic independent in C_{2k} if and only if $|S| \leq 2$ or all vertices do not lie on a path of length less than $k+1$ when $|S| = 3$.*

Proof: We will now show that there are no such sets of cardinality greater than 3. Let $S_3 = \{w_1, w_2, w_3\}$ be a geodetic independent subset of $V(C_{2k})$. Note that $diam(C_{2k}) = k$. So, $d(w_i, w_j) \leq k$. If a fourth vertex w_4 is included in S_3, then w_4 will lie on one of the paths $[w_1, \ldots, w_2]$, $[w_2, \ldots, w_3]$ and $[w_1, \ldots, w_3]$. Let $S_4 = \{w_1, w_2, w_3, w_4\}$ and, without loss of generality, let w_4 be on $w_1, \ldots, w_2$. This means that $I_{C_{2k}}(w_1, w_2) \neq \varnothing$ and $I_{C_{2k}}(S_4) \cap S_4 \neq \varnothing$. It follows that there are no geodetic independent subsets of $V(C_{2k})$ of cardinality 4 and even greater. The converse follows from the definition. □

Theorem 2.5. *Let $k \geq 2$. Then,*

$$g(C_{2k}; x) = \sum_{i=0}^{2} \binom{2k}{i} x^i + \binom{2k}{3} \binom{k-1}{2} x^3.$$

Proof: Let S_3 denote a 3-element geodetic independent subset of $V(C_{2k})$ with $S_3 = \{u_1, u_2, u_3\}$. Consider the graph below:

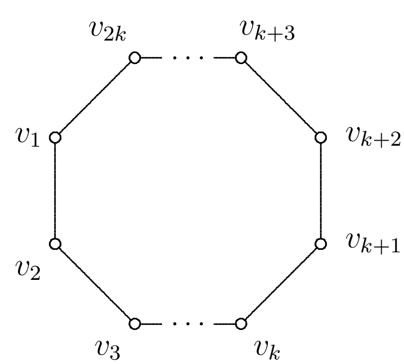

Figure 2: Cycle C_{2k} of order $2k$

Take $u_1 = v_1$ as a fixed point. The choices for the second vertex lie on the path of length $k-1$ with v_1 as one of its endpoints. Without loss of generality, let this path be $v_1, v_2, \ldots, v_k$. If $u_2 = v_2$, count k vertices after v_1 counterclockwise and count k vertices after v_2 clockwise. These vertices form the path $v_{k+2}, v_{k+3}, \ldots, v_{2k}, v_1, \ldots, v_{k+1}$. This means that there is no possible candidate for u_3. By repeating the steps above, we deduce that v_{k+2} is the only possible candidate whenever $u_2 = v_3$, $u_3 \in \{v_{k+2}, v_{k+3}\}$ whenever

$u_2 = v_4$, ..., and $u_3 \in \{v_{k+2}, \ldots, v_{2k-1}\}$ whenever $u_2 = v_k$. This implies that, with fixed vertex $u_1 = v_1$, there are $1 + 2 + \cdots + [(2k-1) - (k+2) + 1] = 1 + 2 + \cdots + (k-2) = \binom{k-1}{2}$ geodetic independent sets by Chuh-Shih-Chieh's Identity. Thus, there are $\binom{2k}{3}\binom{k-1}{2}$ geodetic independent subsets of $V(C_{2k})$ of cardinality 3. Combining gives the result. □

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