

Tensor Product of Banach Spaces and Atomic Solutions of Partial Differential Equations

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Abstract

In this paper, we present a new analytical scheme to generate atomic solutions of partial differential equations. The theory of tensor product in Banach spaces coupled with some features of atomics operators are utilized to attain our results. Some demonstrative examples are given for completeness.

Key words and phrases: Partial differential equations, tensor product, Banach spaces, atomics operator; atomic solution.

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1 Introduction

A Partial Differential Equation (PDE) is an equation involving an unknown function of several variables and its partial derivatives [1, 2]. Many of the foundational theories underlying physics and engineering are expressed by means of partial differential equations [3, 4, 5]. For example, the theory of electromagnetism, including light, can be handled by four partial differential equations called Maxwell's equations. The motion of fluids (including both air and water) can be modeled by the Navier-Stokes system of partial differential equations. Moreover, the continuum mechanics theory, that is used to describe the mechanical behavior of many complex dynamic systems such as granular materials and financial markets, is based on the formulation of partial differential equations [6]. Unlike the theory of ordinary differential equations which centers upon one main theorem that is the existence and uniqueness theorem, it is not easy to master the theory of partial differential equations [7, 8, 9, 10]. There is no general theory known concerning the solvability of all partial differential equations of a given order, even numerically, and classes for which we have general analytic methods of solution are quite restricted [11, 12, 13]. Thus we have to study fairly small classes of partial differential equations individually [14, 15].

The Fourier method, or separation of variables, is one of the most used methods for solving partial differential equations. The concept behind this approach is to reduce the linear partial differential equation to a collection of ordinary differential equations, from which additional solutions can be constructed by generating linear combinations of the original set of solutions, so making use of the notion of superposition [16, 17]. The variable separation method is not without its limits, though. To apply this strategy, a lot of restrictions must be placed on the coefficient expressions and the companion beginning and boundary conditions. It is only useful in extremely rare situations where high degree of symmetry equations are involved. When using the separation of variables method to solve partial differential equations, the solution is typically divided into a product of functions, each of which depends on a single independent variable. In other words, the final solution can be expressed as the product of multiple functions, each of which depends only on one independent variable.

Some of the analytical and numerical techniques of solving partial differential equations are Fourier transform, Laplace transform, Green's functions, finite element method (FEM), finite volume methods (FVM), and finite difference methods (FDM) [18, 19]. In 2010, Khalil [20] introduced a novel

method to solve differential equations for both ordinary and fractional orders. This technique was based on the theory of tensor product of Banach spaces and it can be utilized to obtain the so-called atomic solutions of the differential equation under study. As far as we know, there are a few studies regarding such a technique that were put forward to solve linear and nonlinear 2D partial differential equations. From this point of view, in this paper, we endeavor to introduce a new analytical approach for producing atomic solutions for some kinds of partial differential equations. Before we introduce our main result for atomic solutions of partial differential equations with ordinary orders, we commence with the next definition and theorem [21, 22, 23, 24, 25].

2 Atoms Operators

In this section, we introduce some preliminaries related to the main result of this paper.

Definition 2.1. [22] *Let X and Y be any two Banach spaces and let X^* is the corresponding dual space of X . For $x \in X$ and $y \in Y$, the operator $T : X^* \rightarrow Y$, defined by*

$$T(x^*) = x^*(x)y = \langle x, x^* \rangle y,$$

is a bounded one rank linear operator. We write $x \otimes y$ for T and such operators are called atoms.

Atoms are used in the theory of best approximation in Banach spaces [26, 27] and they are considered among the fundamental ingredients in the theory of tensor product. One of the known results that we need in our paper can be presented in the next theorem [28] which guarantees that if the sum of two atoms is an atom, then either the first components are dependent or the second ones are dependent.

Theorem 2.2. [28] *Let $x_1 \otimes y_1$ and $x_2 \otimes y_2$ be two nonzero atoms in $X \otimes Y$ such that*

$$x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3.$$

Then either $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$.

A nice application of tensor product is the next interesting theorem.

Theorem 2.3. [24] *Let I and J be two compact intervals, and $C(I)$, $C(J)$, and $C(I \times J)$ be the spaces of continuous functions on I , J , and $I \times J$, respectively. Then every $f \in C(I \times J)$ can be written in the form*

$$f(x, y) = \sum_{i=1}^{\infty} u_i(x) v_i(y), \quad (2.1)$$

where $u_i(x) \in C(I)$ and $v_i(y) \in C(J)$.

In this paper, our main object is to obtain an atomic solution that is a solution of the form $u(x, y) = P(x)Q(y)$ to the following partial differential equation

$$u_{xx}(x, y) + u_{xy}(x, y) = f(x)g(y),$$

where u is an unknown function and f, g are given. Clearly, equation (3.2) is inseparable as we cannot move the x -terms to one side and the y -terms to the other. Hence the method of separation of variables does not work.

3 General Scheme for Atomic Solution Method

Consider the following general two-dimensional non-homogeneous linear partial differential equation:

$$u_{xx}(x, y) + u_{xy}(x, y) = f(x)g(y), \quad (3.2)$$

where u is an unknown function and f, g are given and subject to the following conditions:

$$u(0, 0) = 1, \quad u_x(0, 0) = 1, \quad \text{and} \quad u_y(0, 0) = 1. \quad (3.3)$$

According to Theorem 2.2, we start our approach with assuming that

$$u(x, y) = P(x)Q(y). \quad (3.4)$$

Now, we substitute (3.4) into the main partial differential equation (3.2). Hence

$$P''(x)Q(y) + P'(x)Q'(y) = f(x)g(y). \quad (3.5)$$

Clearly, each term of (3.5) is just a product of two functions one of them is pure in x and the other is pure in y . Therefore, in tensor product form, (3.5) can be presented as

$$P''(x) \otimes Q(y) + P'(x) \otimes Q'(y) = f(x) \otimes g(y), \quad (3.6)$$

which implies that the sum of two atoms is an atom. By Theorem 2.3, we have one of the following two cases:

$$\begin{aligned} (i) \quad & P''(x) = P'(x) = f(x), \\ (ii) \quad & Q'(y) = Q(y) = g(y). \end{aligned} \tag{3.7}$$

Case (i) : This case has the following three situations:

$$\begin{aligned} (a) \quad & P''(x) = P'(x), \\ (b) \quad & P''(x) = f(x), \\ (c) \quad & P'(x) = f(x). \end{aligned}$$

From (3.3), without loss of generality, we can assume that

$$P(0) = Q(0) = P'(0) = Q'(0) = 1. \tag{3.8}$$

From situation (a) and conditions (3.8), we get

$$P(x) = e^x. \tag{3.9}$$

Now, by considering (3.9) and both situations (b) and (c), we have $f(x) = e^x$. Thus an atomic solution can be obtained for case (i) provided that $f(x) = e^x$; otherwise, there is no available atomic solution.

The next step is to substitute $P(x) = f(x) = e^x$ into (3.5) which implies

$$Q'(y) + Q(y) = g(y). \tag{3.10}$$

Clearly, (3.10) is a first order linear ordinary differential equation and its general solution can be obtained by multiplying equation (3.10) by the integrating factor; namely, $I = e^y$. Hence

$$Q(y) = e^{-y} \int e^y g(y) dy. \tag{3.11}$$

Thus the first atomic solution with respect to case (i) can be obtained by considering (3.4), (3.9) and (3.11) as follows:

$$u_1(x, y) = e^x \left(e^{-y} \int e^y g(y) dy \right). \tag{3.12}$$

On the other hand, for case (ii), we have the following three situations

$$\begin{aligned} (a^*) \quad & Q'(y) = Q(y), \\ (b^*) \quad & Q'(y) = g(y), \\ (c^*) \quad & Q(y) = g(y). \end{aligned}$$

From situation (a^*) and (3.8), we get

$$Q(y) = e^y. \quad (3.13)$$

Now, by considering (3.13) and both situations (b^*) and (c^*) , we have $g(y) = e^y$. Thus, a second atomic solution can be obtained for case (ii) provided that $g(y) = e^y$; otherwise, there is no available atomic solution for this case. Our next step is to substitute $Q(y) = g(y) = e^y$ into (3.5) which implies

$$P''(x) + P'(x) = f(x). \quad (3.14)$$

Clearly, (3.10) is a second order linear non-homogeneous ordinary differential equation with constant coefficients and its general solution has the form $P(x) = P_h(x) + P_p(x)$, where $P_h(x) = 2 - e^{-x}$ is the complementary solution that can be obtained by considering the companion homogeneous equation; namely, $P''(x) + P'(x) = 0$ together with $P(0) = 1$ and $P'(0) = 1$ (3.8). While the particular solution $P_p(x)$ can be obtained by the method of variation of parameters as follows:

$$\begin{aligned} P_p(y) &= \int \frac{-e^{-x} f(x)}{W[1, e^{-x}]} dx + e^{-x} \int \frac{f(x)}{W[1, e^{-x}]} dx \\ &= \int f(x) dx - e^{-x} \int e^x f(x) dx, \end{aligned} \quad (3.15)$$

where $W[1, e^{-x}]$ is the Wronskian of 1 and e^{-x} . Therefore, the general solution to (3.14) is given by

$$\begin{aligned} P(x) &= P_h(x) + P_p(x) \\ &= 2 - e^{-x} + \int f(x) dx - e^{-x} \int e^x f(x) dx. \end{aligned} \quad (3.16)$$

Hence the second atomic solution with respect to case (ii) can be obtained by considering (3.4), (3.13) and (3.16) as follows:

$$u_2(x, y) = e^y \left(2 - e^{-x} + \int f(x) dx - e^{-x} \int e^x f(x) dx \right). \quad (3.17)$$

4 Applications

In this section, we utilize the new method of atomic solutions to derive two examples for solving partial differential equations for which the separation of variables method does not work. These examples are provided to describe how one can deal with linear and nonlinear 2D partial differential equations, respectively.

Example 1. Consider the following linear 2D partial differential equation

$$u_{xx} + u_{yy} = u_{xy}, \quad (4.18)$$

where $u(x, y)$ is the unknown function and subject to the following conditions:

$$u(0, 0) = 1, \quad u_x(0, 0) = 1, \quad \text{and} \quad u_y(0, 0) = 1. \quad (4.19)$$

By substituting $u(x, y) = P(x)Q(y)$ into (4.18) we get

$$P''(x)Q(y) + P(x)Q''(y) = P'(x)Q'(y). \quad (4.20)$$

Therefore, in tensor product form, (4.20) becomes

$$P''(x) \otimes Q(y) + P(x) \otimes Q''(y) = P'(x) \otimes Q'(y). \quad (4.21)$$

By Theorem 2.3, we have one of the following two cases:

$$\begin{aligned} (i) \quad & P''(x) = P'(x) = P(x), \\ (ii) \quad & Q''(y) = Q'(y) = Q(y). \end{aligned} \quad (4.22)$$

Hence, for case (i), we have the following three situations:

$$\begin{aligned} (a) \quad & P''(x) = P'(x), \\ (b) \quad & P''(x) = P(x), \\ (c) \quad & P'(x) = P(x). \end{aligned}$$

From (3.3), without loss of generality, we can assume that

$$P(0) = Q(0) = P'(0) = Q'(0) = 1. \quad (4.23)$$

From situation (a) and by the conditions (4.23), we get

$$P(x) = e^x. \quad (4.24)$$

Also, both situations (b) and (c), together with conditions from (4.23), give the same result in (4.24). Therefore, an atomic solution exists with respect to case (i). Now, we proceed by substituting (4.24) into (4.20) which implies $Q''(y) - Q'(y) + Q(y) = 0$. This equation and conditions from (4.23) yield

$$Q(y) = \frac{1}{\sqrt{3}} e^{\frac{1}{2}y} \sin\left(\frac{\sqrt{3}}{2}y\right) + e^{\frac{1}{2}y} \cos\left(\frac{\sqrt{3}}{2}y\right). \quad (4.25)$$

Hence, referring to (4.24) and (4.25), the first atomic solution with respect to case (i) is

$$u_1(x, y) = e^x \left[\frac{1}{\sqrt{3}} e^{\frac{1}{2}y} \sin \left(\frac{\sqrt{3}}{2}y \right) + e^{\frac{1}{2}y} \cos \left(\frac{\sqrt{3}}{2}y \right) \right]. \quad (4.26)$$

Similarly, for case (ii), we have the same three situations as those for case (i); namely, $Q''(y) = Q'(y)$, $Q''(y) = Q(y)$, and $Q'(y) = Q(y)$, where $Q(0) = 1$ and $Q'(0) = 1$ (4.23). Hence the second atomic solution with respect to case (ii) is

$$u_2(x, y) = e^y \left[\frac{1}{\sqrt{3}} e^{\frac{1}{2}x} \sin \left(\frac{\sqrt{3}}{2}x \right) + e^{\frac{1}{2}x} \cos \left(\frac{\sqrt{3}}{2}x \right) \right]. \quad (4.27)$$

The two atomic solutions $u_1(x, y)$ (4.26) and $u_2(x, y)$ (4.27) of problem (4.18) are displayed in Figure 1 and Figure 2 respectively.

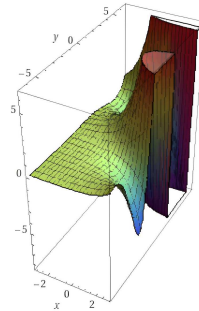


Figure 1: The first atomic solution $u_1(x, y)$ (4.26) of problem (4.18).

Example 2. Consider the following non-linear 2-D partial differential equation

$$u_{xx}u_{yy} + u_xu_y = u_xu, \quad (4.28)$$

where $u(x, y)$ is the unknown function and subject to the following conditions:

$$u(0, 0) = 1, \quad u_x(0, 0) = 1, \quad \text{and} \quad u_y(0, 0) = 1. \quad (4.29)$$

By substituting $u(x, y) = P(x)Q(y)$ into (4.28), we get

$$P''(x)Q''(y) + P'(x)Q'(y) = P'(x)Q(y). \quad (4.30)$$

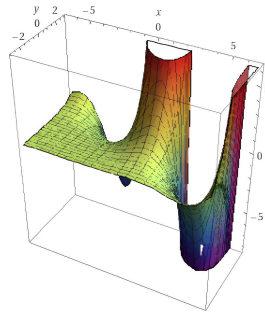


Figure 2: The second atomic solution $u_2(x, y)$ (4.27) of problem (4.18).

Therefore, in tensor product form, (4.30) becomes

$$P''(x) \otimes Q''(y) + P'(x) \otimes Q'(y) = P'(x) \otimes Q(y). \quad (4.31)$$

By Theorem 2.3, the two resultant cases are

$$\begin{aligned} (i) \quad & P''(x) = P'(x) = P(x), \\ (ii) \quad & Q''(y) = Q'(y) = Q(y). \end{aligned} \quad (4.32)$$

From (4.29), without loss of generality, we can assume that

$$P(0) = Q(0) = P'(0) = Q'(0) = 1. \quad (4.33)$$

First, from case (i), we have only one situation that is $P''(x) = P'(x)$. This equation together with the two conditions $P(0) = 1$ and $P'(0) = 1$ (4.33) give

$$P(x) = e^x. \quad (4.34)$$

Now, substituting (4.34) into the main equation (4.30) yields $Q''(y) + Q'(y) - Q(y) = 0$, where $Q(0) = 1$ and $Q'(0) = 1$ (4.33) and hence

$$Q(y) = \frac{1}{2\sqrt{5}} \left[(3 + \sqrt{5}) e^{\left(\frac{-1+\sqrt{5}}{2}\right)y} + (\sqrt{5} - 3) e^{\left(\frac{-1-\sqrt{5}}{2}\right)y} \right]. \quad (4.35)$$

So, the first atomic solution corresponding to case (i) can be obtained by considering (4.34) and (4.35) as follows:

$$u_1(x, y) = \frac{e^x}{2\sqrt{5}} \left[(3 + \sqrt{5}) e^{\left(\frac{-1+\sqrt{5}}{2}\right)y} + (\sqrt{5} - 3) e^{\left(\frac{-1-\sqrt{5}}{2}\right)y} \right]. \quad (4.36)$$

For case (ii), we have the following three situations, namely,

$$\begin{aligned} (a^*) \quad & Q''(y) = Q'(y), \\ (b^*) \quad & Q''(y) = Q(y), \\ (c^*) \quad & Q'(y) = Q(y). \end{aligned}$$

Noting that from the three situation (a*), (b*), and (c*) one can get $Q(x) = e^y$ by assuming the conditions $Q(0) = 1$ and $Q'(0) = 1$ (4.33). Therefore, a second atomic solution exists with respect to case (ii). Now, we substitute $Q(x) = e^y$ into (4.30) which implies $P''(x) = 0$. But, $P(0) = 1$ and $P'(0) = 1$ as in (4.33). So $P(x) = x$ and, as a result, the second atomic solution with respect to case (ii) is

$$u_2(x, y) = xe^y. \quad (4.37)$$

The two atomic solutions $u_1(x, y)$ (4.36) and $u_2(x, y)$ (4.37) of problem (4.28) are displayed in Figure 3 and Figure 4, respectively.

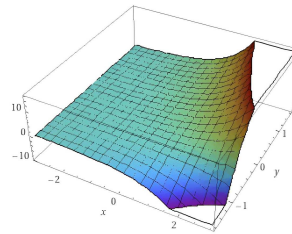


Figure 3: The first atomic solution $u_1(x, y)$ (4.36) of problem (4.28).

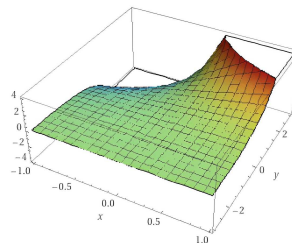


Figure 4: The second atomic solution $u_2(x, y)$ (4.37) of problem (4.28).

5 Conclusions

Through the use of atomic solutions, a new analytical technique for solving non-linear and non-homogeneous partial differential equations has been effectively presented in this work. To arrive at such a notion, the idea of tensor product of Banach spaces along with certain features of atomic operators have been used. It is decided to save several other types of partial differential equations for later research. However, we emphasize the following points:

1. In most cases, the atomic solution approach can give exact solutions to inseparable, non-homogeneous, and non-linear partial differential equations when the method of separation of variables does not work.
2. It is not necessary that each case reported in Theorem 1 admits an atomic solution. This means that the three situations of each case have to provide the same result; otherwise, there is no available atomic solution.

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