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The Union Power Cayley Graph for Cyclic Groups and its Properties

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Abstract

Over the past decades, there has been a lot of studies on graphs associated to groups which are significant in providing effective formulation on the characteristics of the algebraic structures of the groups and graphs. The union power Cayley graph of groups combines the notions of two graphs namely the power graph and Cayley graph. The union power Cayley graph of a group G with respect to the inverse-closed subset S of G, is a graph whose vertices are the elements of G and two vertices, x and y are adjacent if $xy^{-1} \in S$ or if one of them can be written as an integral power of the other. In this manuscript, we present the union power Cayley graphs of the cyclic groups with respect to subsets of size two. Moreover, we classify the union power

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Cayley graphs of cyclic groups as connected, complete, regular and planar. Furthermore, we determine some of the invariants of the union power Cayley graphs of cyclic groups including the clique numbers, vertex chromatic numbers, girths, and diameters.

1 Introduction

In recent times, various approaches have been employed to examine different aspects of a group, including introduction to graphs associated with diverse algebraic structures. In this context, the utilization of the concepts of the groups and their geometric properties to define graphs has emerged as a highly versatile technique in bridging the gap between graph theory and group theory, thereby uncovering the characteristics of graphs related to distinct groups.

In abstract algebra, the integration of group theory into graph theory originated back in the early 18th century when the Cayley graph was being explored. The Cayley graph was first introduced by Cayley [1] in 1988. A Cayley graph of a group G relative to the inverse-closed subset S of G is defined as a graph that has the elements of G as its vertices. Two vertices x and y are adjacent if there exists $s \in S$ such that $x = ys$ or $y = xs$. Several decades later, researchers delved deeper into the concept, employing a variety of group types in their exploration. Wilson [2] claimed that a circulant graph of $Cay(\mathbb{Z}_n, S)$, where $\mathbb{Z}_n$ is the integral cyclic group of order n with certain conditions is unstable. Vilfred [3] asserted that a non-connected circulant graph of $Cay(\mathbb{Z}_n, S)$ consists of $\gcd(S, n)$ copies of the graph $\mathrm{Cay}(\mathbb{Z}_n/\gcd(S, n), S/\gcd(S, n))$. Moreover, Vilfred [4] defined a circulant graph as a Cayley graph or digraph of a cyclic group. In 2013, Marklof [5] studied Cayley graphs of cyclic groups, also known as circulant graphs or multi-loop networks. Moreover, as asserted in [5], the diameter of a random circulant $2k$-regular graph with n vertices scales as $n^{1/k}$.

Various graphs associated with algebraic structures can be employed to identify their unique characteristics. In this context, power graphs have been thoroughly investigated in recent decades. The exploration of directed power graphs was initiated in 2000 by Kelarev and Quinn [6], where they derived certain combinatorial characteristics of the directed power graph for finite groups. Motivated by this, Chakrabarty et al. [7] introduced the undirected power graph $P(G)$ of a group G. Moreover, in [7], it was apparent that $P(G)$

with any finite group G is a connected graph because the identity element of G is adjacent to all other vertices of $P(G)$. More generally, Cameron [8] established that if the power graphs of two finite groups are isomorphic, then their directed power graphs are also isomorphic and showed that the identity and the generators of G make up the set of vertices H of $P(G)$ that are adjacent to every other vertex of $P(G)$ for a finite cyclic group G with non-prime-power order n such that $|H| = 1 + \varphi(n)$, where $\varphi(n)$ is the Euler's function. Mehanian et al. [10] obtained a formula for the power graph of cyclic groups of prime composite power order. In addition, Chelvam and Sattanathan [9] obtained some fundamental characterizations of the power graph for a finite cyclic group. The power graph of a finite group was shown to be perfect, asserted by Alireza *et al.* in [11]. Finally, the explicit formula for the clique number of the power graph of a finite cyclic group was given in [11].

This study is an extension to the concepts of graphs associated with groups. By combining the concept of Cayley graph with power graph, the union power Cayley graph of finite group G related to the subset S of G, $Pow - Cay^{+}(G, S)$, has been introduced in [12]. In this study, we find a generalization of this graph for the cyclic groups with order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, denoted by C_n, where p_i are prime numbers such that $p_1 < p_2 < p_3 < \cdots < p_r$ and $\alpha_i \in \mathbb{N}$. The main focus is to construct $Pow - Cay^{+}(G, S)$ with respect to a subset of $C_n = \langle a \rangle$ of size two, $S^{(2)} = \{a^i, (a^i)^{-1}\}; i \in U(n)$, where $U(n) = \{i : i \in \mathbb{N}, 1 < i < n \text{ and } gcd(i, n) = 1\}$. Various structures of the groups can be discerned by examining this graph along with their characteristics on finite cyclic groups. Hence the general structures for $Pow - Cay^{+}(G, S)$ for the cyclic groups with order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ with some graph-theoretic properties for the graph which include clique numbers, vertex chromatic numbers, girth, and diameters are determined in this study. Moreover, the connectivity, regularity, completeness and planarity of this graph are also analyzed and discussed.

2 Preliminaries

Some basic definitions, notations, and preliminaries are included in this section. In this manuscript, the standard notations used are derived from [13] for groups and [14] for graphs. Besides, all groups considered are finite and the study is limited to cyclic groups C_n of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where p_i are prime numbers such that $p_1 < p_2 < p_3 < \cdots < p_r$, and $\alpha_i \in \mathbb{N}$ with respect to a subset of C_n of size two, $S^{(2)} = \{a^i, (a^i)^{-1}\}; i \in U(n)$, where $U(n) = \{i : i \in \mathbb{N}, 1 < i < n \text{ and } gcd(i,n) = 1\}$.

We consider the undirected simple graph $\Gamma = (V(\Gamma), E(\Gamma))$ without loops or multiple edges, where the set of vertices is denoted by $V(\Gamma)$ and the set of edges is denoted by $E(\Gamma)$. Throughout this paper, we represent the adjacency between vertices a^i and a^j as $a^i \sim a^j$; otherwise, $a^i \nsim a^j$, the number of vertices in the graph Γ as $|V(\Gamma)|$ and the degree of the vertex v as $deg(v)$. A graph Γ is considered a connected graph if there exists a path between every pair of vertices v_i and v_j in $V(\Gamma)$. Otherwise, Γ is called a disconnected graph. A simple graph Γ comprising n vertices is classified as a complete graph of order n, denoted by K_n if every vertex in Γ is connected to all the other vertices. A graph, denoted as Γ, is said to be regular if all its vertices have similar degrees. For a regular graph if the common degree is n, then it is said to be an n-regular graph. A graph Γ is considered to be planar if it can be embedded in a plane in such a way that its edges intersect exclusively at their endpoints. The vertex chromatic number of Γ, represented as $\chi(\Gamma)$, refers to the minimum number of colors needed to color the vertices of Γ such that no two adjacent vertices have the same color. A clique is characterized as a subset X of the set of vertices in the graph Γ, where the subgraph formed by X represents a complete graph. The clique number of a graph Γ, denoted as $\omega(\Gamma)$, is defined as the maximum size of a clique in Γ. In addition, the diameter of a graph Γ, represented as $diam(\Gamma)$, is the maximum distance between any pair of vertices in Γ. The girth in a graph Γ, denoted by $girth(\Gamma)$, is defined as the size of the shortest cycle in Γ. In this manuscript, the disjoint union of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, is denoted as $\Gamma_1 \cup \Gamma_2$. In this case, the vertex set is $V(\Gamma_1 \cup \Gamma_2) = V_1 \cup V_2$, and the edge set is $E(\Gamma_1 \cup \Gamma_2) = E_1 \cup E_2$. Additionally, the joint of $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is denoted as $\Gamma_1 + \Gamma_2$. This graph has a vertex set $V(\Gamma_1 + \Gamma_2) = V_1 \cup V_2$ and an edge set $E(\Gamma_1 + \Gamma_2) = E_1 \cup E_2$ including edges connecting all vertices in V_1 to vertices in V_2. Throughout this manuscript, the cyclic group of order n is represented by C_n. Meanwhile, $\widetilde{C}_n$ denotes the cycle graph, and $\varphi(n) = |U(n)| = |\{i : i \in \mathbb{N}, 1 < i < n \text{ and } gcd(i,n) = 1\}|$ is the Euler's function for $n \in \mathbb{N}$.

First, the formal definition of cyclic group and the union power Cayley graph are provided as follows:

Definition 2.1. [13] *Cyclic Group*
A group G is called a cyclic group if there is an element $a \in G$ that generates

it. More specifically, $G = \langle a \rangle = \{a^n : n \in \mathbb{N}\}$ a cyclic group of order n is denoted as C_n.

Definition 2.2. [12] *Union Power Cayley Graph*
Let G be a group and S be a subset of G such that the identity e of G is not in S, $S \setminus \{e\}$, and $S^{-1} \subseteq S$. Then the union power Cayley graph of G related to S, denoted by $Pow - Cay^{+}(G, S)$, is an undirected simple graph with the vertex set equal to G and two vertices x and y are adjacent if and only if at least one of the following two conditions are satisfied:

1. $xy^{-1} \in S$.
2. $x = y^n$ *or* $y = x^m$ *for some* $m, n \in \mathbb{N}$.

In other words, $E(Pow - Cay^{+}(G, S)) = \{\{x, y\} : \{x, y\} \in E(P(G)) \cup E(Cay(G, S))\}$.

Next, some theorems related to the union power Cayley graph of finite groups are given.

Theorem 2.3. [10] *Let $\mathbb{Z}_n$ be a cyclic group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where p_i are prime numbers such that $p_1 < p_2 < p_3 < \cdots < p_r$ and $\alpha_i \in \mathbb{N}$. Then the power graph, $P(\mathbb{Z}_n)$ with $V(P(\mathbb{Z}_n)) = \{1, 2, \dots, n\}$, $P(\mathbb{Z}_n) = K_{\varphi(n)+1} + \Delta_n \left[K_{\varphi(d_1)}, K_{\varphi(d_2)}, \dots, K_{\varphi(d_m)}\right]$, where Δ_n is a graph with vertex and edge sets $V(\Delta_n) = \{d_i : d_i | n,\ \text{and } d_i \neq 1, d_i \neq n, 1 \leq i \leq m\}$ and $E(\Delta_n) = \{\{d_i, d_j\} : d_i | d_j, 1 \leq i < j \leq m\}$, respectively.*

Theorem 2.4. [15] *Every Cayley graph $Cay(G, S)$ is $|S|$- regular.*

Theorem 2.5. [3] *Let $a_k \in S$. Then in $\widetilde{C}_n(S)$, the length of a cycle of period k is $\frac{n}{\gcd(n,k)}$ and the number of disjoint periodic cycles of period k is* $\gcd(n, k)$.

Proposition 2.6. [3] *Let $Cay(\mathbb{Z}_n, S)$, where $S = \{r_1, r_2, ..., r_k\}$, and $Cay(\mathbb{Z}_n, S) \cong \widetilde{C}_n(\{r_1, r_2, ..., r_k\})$. Then $\widetilde{C}_n(\{r_1, r_2, ..., r_k\})$ is connected if and only if $gcd(n, r_i) = 1$, for $1 \leq r_i \leq k$.*

Theorem 2.7. [11] *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ with $p_1 < p_2 < \cdots < p_m$ and $\alpha_i \in \mathbb{N}$. Then $\omega(P(C_n)) = \varphi(n) + \varphi\left(\frac{n}{p_1}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\alpha_1}}\right) + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2}}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m - 1}}\right) + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}}\right)$,*

where φ is the Euler's function.

Theorem 2.8. [16] *Two graphs generated by two cyclic groups are isomorphic if and only if the cyclic groups are isomorphic.*

3 Results

In this section, we begin by presenting the union power Cayley graph on cyclic groups C_n, of order n, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, and p_i are prime numbers such that $p_1 < p_2 < p_3 < \cdots < p_r$ and $\alpha_i \in \mathbb{N}$ with respect to a subset of C_n of size two, $S^{(2)} = \{a^i, (a^i)^{-1}\}; i \in U(n)$. Next, we provide some propositions related to the graph using the general presentations of the graph. In addition, we use the general presentations to classify the graphs based on the connectivity, completeness, regularity, and planarity of the graphs. The general presentation of this graph is given in the following subsection.

3.1 General Presentations of the Union Power Cayley Graph on Cyclic Groups

In the following theorem, we present the union power Cayley graph for a cyclic group C_n of order n, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, and $\alpha_i \in \mathbb{N}$, with respect to the subsets of size two.

Theorem 3.1. Let $C_n = \langle a \rangle$ be a cyclic group of order n generated by a, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, and p_i are prime numbers such that $p_1 < p_2 < p_3 < \cdots < p_r$, and $\alpha_i \in \mathbb{N}$. Then the union power Cayley graph of C_n with respect to a subset of C_n of size two, $S^{(2)} = \{a^i, (a^i)^{-1}\}; i \in U(n)$, is

$$Pow - Cay^{+}\left(C_n, S^{(2)}\right) \cong K_{\varphi(n)+1} + \nabla_n \left[K_{\varphi(d_1)}, K_{\varphi(d_2)}, \dots, K_{\varphi(d_m)}\right],$$

where ∇_n is a graph with vertex and edge sets
$V(\nabla_n) = \{d_i : d_i | n, \text{ and } d_i \neq 1, d_i \neq n, 1 \leq i \leq m\}$
and
$E(\nabla_n) = \{\{d_i, d_j\} : d_i | d_j \bigvee d_i = a^i d_j; 1 \leq i < j \leq m \text{ and } a^i \in S^{(2)}\}$,
respectively.

Proof. Suppose $C_n = \langle a \rangle$ is a cyclic group of order n, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$. By Definition 2.2, $E(Pow{-}Cay^{+}(C_n, S^{(2)})) = \{\{a^i, a^j\} : \{a^i, a^j\} \in E(P(C_n)) \cup E(Cay(C_n, S^{(2)}))\}$. This means that there is a need to consider edges of $P(C_n)$ and the edges of $Cay(C_n, S^{(2)})$ and find their union. First, by Theorem 2.8, since $\mathbb{Z}_n \cong C_n$, $P(C_n) \cong P(\mathbb{Z}_n)$. Therefore, the general presentation for power graph of C_n given in Theorem 2.3 shows that the power graph $P(C_n)$ has many complete subgraphs isomorphic to
$K_{\varphi(n)+1} + \Delta_n \left[K_{\varphi(d_1)}, K_{\varphi(d_2)}, \dots, K_{\varphi(d_m)}\right]$, where Δ_n is a graph with vertex and edge sets

$V(\Delta_n) = \{d_i : d_i|n, \text{ and } d_i \neq 1, d_i \neq n, 1 \leq i \leq m\}$
and
$E(\Delta_n) = \{\{d_i, d_j\} : d_i|d_j, 1 \leq i < j \leq m\}$,
such that all elements belong to $U(n)$ with an identity element are adjacent to every vertex in the graph and constitute a complete subgraph of order $\varphi(n) + 1$.
Besides, all vertices of degrees d_i and d_j are adjacent for some i and j, which belong to $\{K_{\varphi(d_1)}, K_{\varphi(d_2)}, \ldots, K_{\varphi(d_m)}\}$, where $\{d_1, \ldots, d_m\} = D(n) - \{1, n\}$, where $D(n)$ is the set of all divisors of n and constitute complete subgroups of orders $\varphi(d_i), 1 \leq i \leq m$. Since C_n is a cyclic group, $|d_i|||d_j|$. Also, from Theorem 2.4, $Cay(G, S)$ is $|S|$-regular. Hence $Cay\left(C_n, S^{(2)}\right)$ is a 2-regular graph since $|S^{(2)}| = 2$. Moreover, there is an edge joining the vertex a^i and a^j in C_n if and only if $a^i(a^j)^{-1} = g$ for $g \in S^{(2)}$. For a set $S^{(2)} = \{a^i, (a^i)^{-1}\}, i \in U(n)$, the $\gcd(n, i) = 1$. By Theorems 2.8 and 2.5 and Proposition 2.6, the Cayley graph with respect to $S^{(2)}$, a subset of C_n of size two, is a connected circulant graph and consists of $\gcd(n, i)$ copies of the graph. In other words, there is one connected circulant graph. Since $|C_n| = n$ and $\gcd(n, i) = 1$, $Cay(C_n, S^{(2)})$ is isomorphic to a cycle graph of order n, $\widetilde{C}_n$. Secondly, there is a need to check for the common edges as a result of taking the union of the edges of $Cay(C_n, S^{(2)})$ with $P(C_n)$, if the all edges belong to $Cay(C_n, S^{(2)})$ in $P(C_n)$ or if there are some additional new edges. To determine this, recall that $g^i \sim g^j$ if $g^i = a^i g^j; a^i \in S^{(2)}$. Thus $Cay(C_n, S^{(2)}) = \widetilde{C}_n$ which means that $g^i \sim g^{i+j}$, where $j \in U(n)$. Hence $d_i = a^i d_j$ are the new edges formed by the impact of taking union of edges with the Cayley graph. These edges linked some vertices of $K_{\varphi(d_i)}$ and $K_{\varphi(d_j)}$ for $i \neq j$ besides edges of power graph which make the edges $E(\Delta_n) = \{\{d_i d_j\} : d_i|d_j \bigvee d_i = a^i d_j; 1 \leq i < j \leq m; a^i \in S^{(2)}\}$. It remains to show that these new edges do not from new cliques. This can be observed easily, since $g^i \sim g^{i+j}$, for all $g^i \in C_n$ and $j \in S^{(2)}$, when $j \notin U(C_n)$ and $j \neq e$. Hence these new edges link some vertices of $K_{\varphi(d_i)}$ with some vertices of $K_{\varphi(d_j)}$ for $i \neq j$ which means that they are not linked to all the vertices of the subgraphs. Hence they did not make new cliques but only create some new edges. Therefore, these new edges do not form a new clique in $Pow - Cay^+\left(C_n, \quad S^{(2)}\right)$.
Hence $Pow - Cay^+\left(C_n, S^{(2)}\right) \cong K_{\varphi(n)+1} + \nabla_n\left[K_{\varphi(d_1)}, K_{\varphi(d_2)}, \ldots, K_{\varphi(d_m)}\right]$, where ∇_n is a graph with vertex and edge sets
$V(\nabla_n) = \{d_i : d_i|n, \text{ and } d_i \neq 1, d_i \neq n, 1 \leq i \leq m\}$
and
$E(\nabla_n) = \{\{d_i, d_j\} : d_i|d_j \bigvee d_i = a^i d_j; 1 \leq i < j \leq m \text{ and } a^i \in S^{(2)}\}$.

□

In the next subsection, we present the invariants of the union power Cayley graphs for a cyclic group of order n, C_n, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ with respect to the subset of size two. These invariants comprise the clique number, chromatic numbers, diameter, and girth of the graph.

3.2 Invariants of the Union Power Cayley Graphs Associated to Cyclic Group of Order n with Subsets of Size Two

In this subsection, some of the invariants of the union power Cayley graphs for a cyclic group of order, n, with respect to the subsets of size two are investigated, comprising the clique number, chromatic numbers, diameter, and girth of the graph.

In the following proposition, we find the clique number of the union power Cayley graphs for a cyclic group of order, n, with respect to the subsets of size two.

Proposition 3.2. Let $C_n = \langle a \rangle$ be a cyclic group of order n generated by a, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, and p_i are prime numbers such that $p_1 < p_2 < p_3 < \dots < p_r$, and $\alpha_i \in \mathbb{N}$. Then the clique number of the union power Cayley graphs of C_n with respect to a subset of C_n of size two, $S^{(2)} = \{a^i, (a^i)^{-1}\}; i \in U(n)$, is

$$\omega\left(Pow - Cay^+\left(C_n, S^{(2)}\right)\right) = \varphi(n) + \varphi\left(\frac{n}{p_1}\right) + \dots + \varphi\left(\frac{n}{p_1^{\alpha_1}}\right) + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2}\right) + \dots + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2}}\right) + \dots + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m}\right) + \dots + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m - 1}}\right) + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}}\right),$$

where φ is the Euler's function.

Proof. Let C_n be a cyclic group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$. By Theorem 3.1, $Pow - Cay^+\left(C_n, S^{(2)}\right) \cong K_{\varphi(n)+1} + \nabla_n \left[K_{\varphi(d_1)}, K_{\varphi(d_2)}, \dots, K_{\varphi(d_m)}\right]$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ be a natural number and $m = \alpha_1 + \alpha_1 + \dots + \alpha_m$. Also, let p be the set of all m-tuples $(d_1, d_2, \dots, d_m)$ where $d_1 > d_2 > \dots > d_m$ is the chain of divisors of n and $(d_i - 1)/d_i$ be a prime for all $i = 1, 2, 3, \dots, m$. Let X be one of the cliques of $Pow - Cay^+\left(C_n, S^{(2)}\right)$. It can easily be observed that if $x \in V(X)$, then $r \in V(X)$ when $gcd(r, |x|) = 1$. Thus the elements $V(X)$ can be partitioned into sets which contains elements of the same order. That is, $V(X)$ can be written as disjoint union of cliques $V(X) = X_{h_1} \cup X_{h_2} \cup \dots \cup X_{h_k}$, where h_i posses all of the elements of order h_k and $|X_{h_i}| = \varphi(h_i)$. This means that a function f can be defined as $f : P \to \mathbb{N}$ by $f(d_1, d_2, ..., d_m) = \varphi(d_1)+\varphi(d_2)+...+\varphi(d_m)$. Recall that, $E(\nabla_n) = \{d_i d_j : d_i|d_j \bigvee d_i = a^i d_j; 1 \leq i < j \leq m$ and $i \in U(n)\}$ which shows that there are some new edges formed from the Cayley graph as a result of taking edge union. However, since the edges linked only some vertices of $K_{\varphi(d_i)}$, for $i \neq j$, the new edges do not produce new cliques. Hence the new edges do not change the size of the maximum clique of the graph. Thus $\omega\left(Pow - Cay^+\left(C_n, S^{(2)}\right)\right) = \omega(P(C_n))$. To see this, since $x \sim y$ if $\langle x\rangle \subseteq \langle y\rangle$ or $\langle y\rangle \subseteq \langle x\rangle$, $d_1, d_2, ..., d_k$ is a chain of positive divisors of n such that $f_1|f_2|f_3|\cdots|f_k$. Then a clique of size $\sum_{i=0}^{k}\varphi(f_i)$ in $Pow - Cay^+\left(C_n, S^{(2)}\right)$ can be found. Suppose that $V(Y) = Y_{d_0} \cup Y_{d_1} \cup ... \cup Y_{d_m}$, where $d_0 = n$ and Y_{d_i} posses all elements of order d_i, $|Y_{d_i}| = \varphi(d_i)$. Since Y is a maximal clique, $|V(Y)| = \sum_{i=0}^{n}\varphi(d_i) = \omega(P(C_n))$. From the above step and Theorem 2.7 ,$\omega(P(C_n)) = \varphi(n) + \varphi\left(\frac{n}{p_1}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}}\right)+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m-1}}\right)+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}}\right)$. Since $\omega\left(Pow - Cay^+\left(C_n, S^{(2)}\right)\right) = \omega(P(C_n))$, $\omega\left(Pow - Cay^+\left(C_n, S^{(2)}\right)\right) = \varphi(n) + \varphi\left(\frac{n}{p_1}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\alpha_1}}\right) + \varphi\left(\frac{n}{p_1^{\alpha_1}p_2}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m-1}}\right)+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}}\right)$. □

Proposition 3.3. Let $C_n = \langle a\rangle$ be a cyclic group of order n generated by a, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$, and let p_i be prime numbers such that $p_1 < p_2 < p_3 < \cdots < p_r$, and $\alpha_i \in \mathbb{N}$. Then the chromatic number of the union power Cayley graph of C_n with respect to a subset of C_n of size two, $S^{(2)} = \{a^i, (a^i)^{-1}\}; i \in U(n)$, is

$$\chi\left(Pow - Cay^+\left(C_n, S^{(2)}\right)\right) = \varphi(n)+\varphi\left(\frac{n}{p_1}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}}\right)+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m-1}}\right)+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}}\right),$$

where φ is the Euler's function.

Proof. Suppose C_n is a cyclic group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m}$. By Theorem 3.1, $Pow - Cay^+\left(C_n, S^{(2)}\right) \cong K_{\varphi(n)+1} + \nabla_n\left[K_{\varphi(d_1)}, K_{\varphi(d_2)}, \dots, K_{\varphi(d_m)}\right]$. Even though ∇_n creates some new vertices that link some of the vertices of $K_{\varphi(d_i)}$ and $K_{\varphi(d_j)}$ for $i \neq j$, some vertices of $K_{\varphi(d_i)}$ are not joined with some vertices of $K_{\varphi(d_j)}$. Let $u \in V(K_{\varphi(d_i)})$ and $v \in V(K_{\varphi(d_j)})$ such that $u \nsim v$. Then u and v can be assigned with a single color. This means that some colors can be shared among some vertices. Thus the colors of non-adjacent vertices $V(K_{\varphi(d_i)})$ and $V(K_{\varphi(d_j)})$ for $i \neq j$ can be shared from the vertices of one clique to the vertices of another distinct clique. By considering the above fact and the fact that each vertex of the maximum clique must be colored distinctly, the minimum number of colors required to properly color the vertices of the graph is its clique number. By Proposition 3.2, $\chi\left(Pow-Cay^{+}\left(C_n, S^{(2)}\right)\right) = \varphi(n)+\varphi\left(\frac{n}{p_1}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}}\right)+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m}\right)+\cdots+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m-1}}\right)+\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}}\right)$. □

Proposition 3.4. Let $C_n = \langle a \rangle$ be a cyclic group of order n generated by a, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}$, and let p_i be prime numbers such that $p_1 < p_2 < p_3 < \cdots < p_r$, and $\alpha_i \in \mathbb{N}$. Then the diameter of the union power Cayley graph of C_n with respect to a subset of C_n of size two, $S^{(2)} = \{a^i, (a^i)^{-1}\}\,; i \in U(n)$, is

$$diam\left(Pow-Cay^{+}\left(C_n, S^{(2)}\right)\right) = 2.$$

Proof. Let C_n be a cyclic group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_m^{\alpha_m}$, C_n. By Theorem 3.1, $Pow-Cay^{+}\left(C_n, S^{(2)}\right) \cong K_{\varphi(n)+1} + \nabla_n\left[K_{\varphi(d_1)}K_{\varphi(d_2)}, \ldots, K_{\varphi(d_m)}\right]$. The general presentation of the graph shows that $V(K_{\varphi(n+1)})$ are adjacent to all the vertices of the graph, but not all the vertices of $K_{\varphi(d_i)}$ are adjacent to all the vertices of $K_{\varphi(d_j)}$ for $i \neq j$. Let $u \in V(K_{\varphi(n+1)})$ and pick arbitrary vertices $v_1 \in V(K_{\varphi(d_i)})$ and $v_2 \in V(K_{\varphi(d_j)})$ such that $v_1 \nsim v_2$. Then v_2 can be reachable by v_1 through u, which is a walk of length 2 as the maximum distance to reach any pair of vertices of the graph. Hence $diam\left(Pow-Cay^{+}\left(C_n, S^{(2)}\right)\right) = 2$. □

Proposition 3.5. Let $C_n = \langle a \rangle$ be a cyclic group of order n generated by a, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}$, and p_i are prime numbers such that $p_1 < p_2 < p_3 < \cdots < p_r$, and $\alpha_i \in \mathbb{N}$. Then the girth of the union power Cayley graph of C_n with respect to a subset of C_n of size two, $S^{(2)} = \{a^i, (a^i)^{-1}\}\,; i \in U(n)$ is,

$$girth\left(Pow-Cay^{+}\left(C_n, S^{(2)}\right)\right) = 3.$$

Proof. Let C_n be a cyclic group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_m^{\alpha_m}$. Then by Theorem 3.1, $Pow-Cay^{+}\left(C_n, S^{(2)}\right) \cong K_{\varphi(n)+1} + \nabla_n\left[K_{\varphi(d_1)}K_{\varphi(d_2)}, \ldots, K_{\varphi(d_m)}\right]$. Suppose n is the product of the two smallest prime, $n \geq 6$. Thus $|V(K_{\varphi(n+1)})| \geq 3$ which shows that, for all n, the graph $Pow-Cay^{+}\left(C_n, S^{(2)}\right)$ contains a minimum cycle of length 3. Therefore, $girth\left(Pow - Cay^{+}\left(C_n, S^{(2)}\right)\right) = 3$. ☐

The general presentation of the union power Cayley graphs for a cyclic group of order n, C_n, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ with respect to the subsets of size two in Subsection 3.1, and the invariants in Subsection 3.2 are used to classify the graph as one of connected, complete, regular, or planar, which are presented in the following Subsection.

3.3 Classification of the Union Power Cayley Graphs Associated to Cyclic Group of Order n With Respect to the Subsets of Size Two

In the next proposition, we give the connectivity, regularity, completeness, and planarity of the union power Cayley graphs for a cyclic group of order n, C_n, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ with respect to the subsets of size two.

Proposition 3.6. Let $C_n = \langle a \rangle$ be a cyclic group of order n generated by a, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$. Suppose p_i are prime numbers such that $p_1 < p_2 < p_3 < \cdots < p_r$ and $\alpha_i \in \mathbb{N}$. Then the union power Cayley graph of C_n with respect to a subset of C_n of size two, $S^{(2)} = \left\{a^i, (a^i)^{-1}\right\}; i \in U(n)$ is connected and not regular, hence not complete.

Proof. Let C_n be a cyclic group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$. By Theorem 3.1, $Pow - Cay^{+}\left(C_n, S^{(2)}\right) \cong K_{\varphi(n)+1} + \nabla_n\left[K_{\varphi(d_1)}, K_{\varphi(d_2)}, \dots, K_{\varphi(d_m)}\right]$. By the graph presentation, $K_{\varphi(n)+1}$ is a joint with $\left[K_{\varphi(d_1)}, K_{\varphi(d_2)}, \dots, K_{\varphi(d_m)}\right]$. This means that all the vertices of the graph can be reachable from one another through $V(K_{\varphi(n)+1})$. Therefore, $Pow - Cay^{+}\left(C_n, S^{(2)}\right)$ is connected. For regularity, since $d_i | n$, $\varphi(d_i) | \varphi(n)$ which implies that $deg(K_{\varphi(n)+1}) > deg(K_{\varphi(d_i)})$. Therefore, the degree of some vertices of the graph differs. Hence $Pow - Cay^{+}\left(C_n, S^{(2)}\right)$ is not regular. For completeness, since not all vertices of $K_{\varphi(d_i)}$ are adjacent with all the vertices of $K_{\varphi(d_j)}$ for $i \neq j$, $Pow - Cay^{+}\left(C_n, S^{(2)}\right)$ is not complete. ☐

In the next proposition, we give the planarity of the union power Cayley graphs for a cyclic group of order, $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, with respect to the subsets of size two.

Proposition 3.7. Let $C_n = \langle a \rangle$ be a cyclic group of order n generated by a, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ and let p_i be prime numbers such that $p_1 < p_2 <$

$p_3 < \cdots < p_r$ and $\alpha_i \in \mathbb{N}$. Then the union power Cayley graph of C_n with respect to a subset of C_n of size two, $S^{(2)} = \{a^i, (a^i)^{-1}\}; i \in U(n)$ is not planar.

Proof. Let C_n be a cyclic group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$. By Theorem 3.1, $Pow-Cay^+\left(C_n, S^{(2)}\right) \cong K_{\varphi(n)+1} + \nabla_n\left[K_{\varphi(d_1)} K_{\varphi(d_2)}, \dots, K_{\varphi(d_m)}\right]$ and by Proposition 3.2, $\omega\left(Pow - Cay^+\left(C_n, S^{(2)}\right)\right) = \varphi(n) + \varphi\left(\frac{n}{p_1}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\alpha_1}}\right) + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2}}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m - 1}}\right) + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}}\right)$. Suppose n is the product of the two smallest prime; that is $n = 6$. Then $\omega\left(Pow - Cay^+\left(C_n, S^{(2)}\right)\right)$ will be greater than four which cannot be drawn in a plane without edge crossing. Hence $Pow-Cay^+\left(C_n, S^{(2)}\right)$ is not planar. $\square$

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