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# A new uniform bound on Poisson approximation

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**Abstract**

In this study, the stein-Chen method is used to determine a new uniform bound on Poisson approximation.

## 1 Introduction and main result

We start with the context of the Stein-Chen method for Poisson approximation that was first introduced in [1]. The method was inspired by the Stein's equation for the Poisson distribution with mean $\lambda > 0$ and, for given $h$, the equation is of the form

$$h(x) - \mathcal{P}_\lambda(h) = \lambda g(x+1) - xg(x), \tag{1.1}$$

where $\mathcal{P}_\lambda(h) = e^{-\lambda} \sum_{l=0}^{\infty} h(l) \frac{\lambda^l}{l!}$ and $g$ and $h$ are bounded real-valued function defined on $\mathbb{N} \cup \{0\}$.

For $A \subseteq \mathbb{N} \cup \{0\}$, let $h_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

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Let $x \in \mathbb{N} \cup \{0\}$ and $C_x = \{0, ..., x\}$. The solution $g_A$ of (1.1) can be expressed as [2]:

$$g_A(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}\{\mathcal{P}_\lambda(h_{A\cap C_{x-1}}) - \mathcal{P}_\lambda(h_A)\mathcal{P}_\lambda(h_{C_{x-1}})\} & \text{if } x \geq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Similarly, for $A = C_{x_0}$ where $x_0 \in \mathbb{N} \cup \{0\}$, $g_{C_{x_0}}$ can be written as

$$g_{C_{x_0}}(x) = \begin{cases} (x-1)!\lambda^{-w}e^{\lambda}\{\mathcal{P}_\lambda(h_{C_{x-1}})\mathcal{P}_\lambda(1-h_{C_{x_0}})\} & \text{if } x \leq x_0 \\ (x-1)!\lambda^{-w}e^{\lambda}\{\mathcal{P}_\lambda(h_{C_{x_0}})\mathcal{P}_\lambda(1-h_{C_{x-1}})\} & \text{if } x > x_0 \\ 0 & \text{if } x = 0, \end{cases} \tag{1.2}$$

and observe that $g_{C_{x_0}}(x) > 0$ for every $x \in \mathbb{N}$.

For $x_0 \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{N}$, let $\Delta g_{x_0}(x) = g_{x_0}(x+1) - g_{x_0}(x)$ and $\Delta g_{C_{x_0}}(x) = g_{C_{x_0}}(x+1) - g_{C_{x_0}}(x)$. Barbour et al. [2] gave a uniform bound for the suppremum of $|g_{C_{x_0}}(x)|$,

$$\sup_{x_0 \geq 0} |\Delta g_{C_{x_0}}(x)| \leq \lambda^{-1}(1-e^{-\lambda}). \tag{1.3}$$

In this paper, we focus our attention on determining a new uniform bound, with respect to the result in (1.3), which can be stated as the following theorem:

**Theorem 1.1.** *For $x_0 \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{N}$, we have the following:*

$$\sup_{x_0 \geq 0} |\Delta g_{C_{x_0}}(x)| \leq \begin{cases} \dfrac{1}{2} & \text{if } 0 < \lambda < 1, \\ \max\left\{\dfrac{e^{-\lambda}+\lambda-1}{\lambda^2}, \min\left\{\dfrac{1-e^{-\lambda}}{\lambda}, \dfrac{1}{3}\right\}\right\} & \text{if } \lambda \geq 1. \end{cases}$$

## 2 Proof of main result

The following lemmas are used to prove the main result:

**Lemma 2.1.** *For $x_0, x \in \mathbb{N}$, we have the following:*

1. *$\Delta g_{C_{x_0}}(x)$ is increasing function for $x > x_0$.*

2. *$|\Delta g_{C_{x_0}}(x)| \leq \lambda^{-1} \min\left\{1-e^{-\lambda}, \frac{\lambda}{\varphi(x_0)}\right\}$,*

*where* $\varphi(x_0) = \begin{cases} x_0 + 2 & \text{if } x_0 \leq \lambda, \\ x_0 + 1 & \text{if } x_0 > \lambda. \end{cases}$

**Proof**. 1. We have to show that

$$\Delta g_{C_{x_0}}(x+1) - \Delta g_{C_{x_0}}(x) > 0$$

for $x \in \{x_0+1, x_0+2, ...\}$.

By [3], we have

$$\begin{aligned}
&\Delta g_{C_{x_0}}(x+1) - \Delta g_{C_{x_0}}(x) \\
&= (x-1)!\lambda^{-(x+2)}\mathcal{P}_\lambda(h_{C_{x_0}}) \\
&\quad \times \left\{ x\left[\sum_{k=x+2}^{\infty}(x+1)\frac{\lambda^k}{k!} - \sum_{k=x+1}^{\infty}\frac{\lambda^{k+1}}{k!}\right] - \lambda\left[\sum_{k=x+1}^{\infty}x\frac{\lambda^k}{k!} - \sum_{k=x}^{\infty}\frac{\lambda^{k+1}}{k!}\right]\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
&x\left\{\sum_{k=x+2}^{\infty}(x+1)\frac{\lambda^k}{k!} - \sum_{k=x+1}^{\infty}\frac{\lambda^{k+1}}{k!}\right\} - \lambda\left\{\sum_{k=x+1}^{\infty}x\frac{\lambda^k}{k!} - \sum_{k=x}^{\infty}\frac{\lambda^{k+1}}{k!}\right\} \\
&= \sum_{k=x+2}^{\infty}x(x+1-k)\frac{\lambda^k}{k!} - \sum_{k=x+2}^{\infty}(x+1-k)k\frac{\lambda^k}{k!} \\
&= \sum_{k=x+2}^{\infty}(x-k)(x+1-k)\frac{\lambda^k}{k!} \\
&> 0,
\end{aligned}$$

we obtain $\Delta g_{C_{x_0}}(x+1) - \Delta g_{C_{x_0}}(x) > 0$.

Therefore, $\Delta g_{C_{x_0}}(x)$ is increasing for $x > x_0$.

2. Consider the case $x_0 > \lambda$.

By [4], it suffices to show the case of $x_0 \leq \lambda$; that is,

$$|\Delta g_{C_{x_0}}(x)| \leq \lambda^{-1} \min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0+2}\right\}.$$

Since $\Delta g_{C_{x_0}}(x)$ is an increasing function for $x > x_0$ and $\Delta g_{C_{x_0}}(x) < 0$,

$$\begin{aligned}
&\left|\Delta g_{C_{x_0}}(x)\right| \leq -\Delta g_{C_{x_0}}(x_0+1) \\
&= \Delta g_{\{x_0+1\}}(x_0+1) - \Delta g_{C_{x_0+1}}(x_0+1) \\
&= \frac{1}{\lambda}\mathcal{P}_\lambda(1-h_{C_{x_0+1}}) + \frac{1}{x_0+1}\mathcal{P}_\lambda(h_{C_{x_0}}) \\
&\quad - x_0!\lambda^{-(x_0+2)}\left[e^{-\lambda}\sum_{j=x_0+2}^{\infty}\frac{\lambda^j}{j!}\right]\left[\sum_{k=0}^{x_0}(x_0+1-k)\frac{\lambda^k}{k!}\right] \\
&= \frac{1}{\lambda}\mathcal{P}_\lambda(1-h_{C_{x_0+1}}) + \frac{1}{x_0+1}\mathcal{P}_\lambda(h_{C_{x_0}}) \\
&\quad - x_0!e^{-\lambda}\left[\frac{1}{(x_0+2)!} + \frac{\lambda}{(x_0+3)!} + \dots\right]\left[\sum_{k=0}^{x_0}(x_0+1-k)\frac{\lambda^k}{k!}\right] \\
&\leq \sum_{k=x_0+2}^{\infty} e^{-\lambda}\frac{\lambda^{k-1}}{k(k-1)!} + \frac{1}{x_0+2}\mathcal{P}_\lambda(h_{C_{x_0}}) \\
&\leq \frac{1}{x_0+2}\sum_{k=x_0+1}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} + \frac{1}{x_0+2}\sum_{k=0}^{x_0} e^{-\lambda}\frac{\lambda^k}{k!} \\
&= \frac{1}{x_0+2}.
\end{aligned}$$

By [4], for $x \le x_0$, we have

$$
\begin{aligned}
&\left|\Delta g_{C_{x_0}}(x)\right| \\
&\le \Delta g_{C_{x_0}}(x_0) \\
&= \Delta g_{C_{x_0-1}}(x_0) + \Delta g_{\{x_0\}}(x_0) \\
&= (x_0-1)!\lambda^{-(x_0+1)} e^{\lambda} \mathcal{P}_\lambda(h_{C_{x_0-1}}) \left[ x_0 \sum_{k=x_0+1}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} - \lambda \sum_{k=x_0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \right] \\
&\quad + \frac{1}{x_0} \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=x_0+1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{k!} \\
&= (x_0-1)!\lambda^{-(x_0+1)} \mathcal{P}_\lambda(h_{C_{x_0-1}}) \left[ \sum_{k=x_0+1}^{\infty} \frac{x_0\lambda^k}{k!} - \sum_{k=x_0+1}^{\infty} \frac{k\lambda^k}{k!} \right] \\
&\quad + \frac{1}{x_0} \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=x_0+1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{k!} \\
&= -(x_0-1)! \mathcal{P}_\lambda(h_{C_{x_0-1}}) \left[ \sum_{k=x_0+1}^{\infty} \frac{(k-x_0)\lambda^{k-(x_0+1)}}{k!} \right] \\
&\quad + \frac{1}{x_0} \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=x_0+1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{k!} \\
&= \left[ -\frac{1}{x_0(x_0+1)} - \frac{2\lambda}{x_0(x_0+1)(x_0+2)} - \cdots \right] \mathcal{P}_\lambda(h_{C_{x_0-1}}) \\
&+ \frac{1}{x_0} \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=x_0+1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{k!} \\
&\le \left[ -\frac{1}{x_0(x_0+1)} - \frac{2\lambda}{x_0(x_0+1)(x_0+2)} \right] \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} \\
&\quad + \frac{1}{x_0} \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=x_0+1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{k!} \\
&= \left[ \frac{1}{x_0} - \frac{1}{x_0(x_0+1)} \right] \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} \\
&\quad + \sum_{k=x_0+1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{k!} - \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} \left[ \frac{2\lambda}{x_0(x_0+1)(x_0+2)} \right] \\
&= \frac{1}{x_0+1} \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=x_0+1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{k(k-1)!} - \sum_{k=0}^{x_0-1} \frac{e^{-\lambda}\lambda^k}{k!} \left[ \frac{2\lambda}{x_0(x_0+1)(x_0+2)} \right]
\end{aligned}
$$

$$= \frac{1}{x_0+1}\sum_{k=0}^{x_0-1}\frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=x_0}^{\infty}\frac{e^{-\lambda}\lambda^k}{(k+1)k!} - \sum_{k=0}^{x_0-1}\frac{e^{-\lambda}\lambda^k}{k!}\left[\frac{2\lambda}{x_0(x_0+1)(x_0+2)}\right]$$

$$\leq \frac{1}{x_0+1}\sum_{k=0}^{x_0}\frac{e^{-\lambda}\lambda^k}{k!} + \frac{1}{x_0+2}\sum_{k=x_0+1}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!} - \sum_{k=0}^{x_0-1}\frac{e^{-\lambda}\lambda^k}{k!}\left[\frac{2\lambda}{x_0(x_0+1)(x_0+2)}\right]$$

$$= \frac{1}{x_0+2} + \left(\frac{1}{x_0+1} - \frac{1}{x_0+2}\right)\sum_{k=0}^{x_0}\frac{e^{-\lambda}\lambda^k}{k!} - \sum_{k=0}^{x_0-1}\frac{e^{-\lambda}\lambda^k}{k!}\left[\frac{2\lambda}{x_0(x_0+1)(x_0+2)}\right]$$

$$= \frac{1}{x_0+2} + \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0}\frac{e^{-\lambda}\lambda^k}{k!} - \sum_{k=0}^{x_0-1}\frac{e^{-\lambda}\lambda^k}{k!}\left[\frac{\lambda}{x_0(x_0+1)(x_0+2)}\right]$$

$$- \sum_{k=0}^{x_0-1}\frac{e^{-\lambda}\lambda^k}{k!}\left[\frac{\lambda}{x_0(x_0+1)(x_0+2)}\right]$$

$$\leq \frac{1}{x_0+2} + \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0}\frac{e^{-\lambda}\lambda^k}{k!} - \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0-1}\frac{e^{-\lambda}\lambda^k}{k!}$$

$$- \sum_{k=0}^{x_0-1}\frac{(k+1)e^{-\lambda}\lambda^{k+1}}{(k+1)k!}\left[\frac{1}{x_0(x_0+1)(x_0+2)}\right]$$

$$= \frac{1}{x_0+2} + \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0}\frac{e^{-\lambda}\lambda^k}{k!} - \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0}\frac{ke^{-\lambda}\lambda^k}{x_0 k!}$$

$$- \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0-1}\frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \frac{1}{x_0+2} + \frac{1}{(x_0+1)(x_0+2)}\left[\sum_{k=0}^{x_0}\left(1-\frac{k}{x_0}\right)\frac{e^{-\lambda}\lambda^k}{k!}\right]$$

$$- \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0-1}\frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \frac{1}{x_0+2} + \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0-1}\left(\frac{x_0-k}{x_0} - 1\right)\frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \frac{1}{x_0+2} + \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0-1}\left(\frac{-k}{x_0}\right)\frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \frac{1}{x_0+2} - \frac{1}{(x_0+1)(x_0+2)}\sum_{k=0}^{x_0-1}\left(\frac{k}{x_0}\right)\frac{e^{-\lambda}\lambda^k}{k!}$$

$$\leq \frac{1}{x_0+2}.$$

Hence, from the proof mentioned above, we have

$$|\Delta g_{C_{x_0}}(x)| \leq \lambda^{-1} \min\left\{1 - e^{-\lambda}, \tfrac{\lambda}{x_0+2}\right\},$$

which gives the desired result. □

**Proof of Theorem 1.1**. For $0 < \lambda < 1$ and $x_0 = 0$, by [3] we have $|\Delta g_{C_{x_0}}(x)| \leq \lambda^{-2}(e^{-\lambda} + \lambda - 1) \leq \dfrac{1}{2}$. For $x_0 > 0$, by [4] we also obtain $|\Delta g_{C_{x_0}}(x)| \leq \lambda^{-1} \min\left\{1 - e^{-\lambda}, \tfrac{\lambda}{x_0+1}\right\} \leq \dfrac{1}{2}$, which gives the first inequality. For $\lambda \geq 1$ and $x_0 = 0$, we have $|\Delta g_{C_{x_0}}(x)| \leq \lambda^{-2}(e^{-\lambda} + \lambda - 1)$. For $x_0 > 0$, by Lemma 2.1 (2), we have $|\Delta g_{C_{x_0}}(x)| \leq \min\left\{\dfrac{1 - e^{-\lambda}}{\lambda}, \dfrac{1}{3}\right\}$, which yields the second inequality. Consequently, the theorem follows. □

# References


[1] L. H. Y. Chen, Poisson approximation for dependent trials, Ann. Probab., **3,** (1975), 534–545.

[2] A. D. Barbour, L. Holst, S. Janson, Poisson Approximation, Oxford Studies in Probability 2, 1992.

[3] K. Teerapabolarn, K. Neammanee, A non-uniform bound in somatic cell hybrid model, Math. BioSci., **195**, (2005), 56–64.

[4] P. Kongkam, K. Teerapabolarn, A non-uniform bound on Poisson approximation for a sum of independent non-negative integer-valued random variables, Int. J. Math. Comput. Sci., **18**, no. 3, (2023), 557–562.