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Highest endomorphisms of a Boolean lattice

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Abstract

An endomorphism of a finite algebra is said to be *highest* if its pre-period is greater than or equal to the pre-period of all its endomorphisms. In this paper, we characterize all highest endomorphisms of a Boolean lattice.

1 Introduction

One of the significant algebras is a monounary algebra which consists of a set and a unary operation on it. The advantage is its easy visualization. The important theories of unary and monounary algebras are shown in many monographs; for instance, [7, 9, 10, 11].

Let $f : A \to A$ be a unary function on a set A. An element $a \in A$ is called a *cyclic* if $f^n(a) = a$ for some $n \in \mathbb{N}$. The *height* of an element $x \in A$,

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denoted by $\mathrm{ht}(x)$, is the least non-negative integer i such that $f^i(x)$ is a cyclic element. The *height* of the finite monounary algebra (A, f) is defined by

$$\mathrm{ht}(A, f) := \max \{\mathrm{ht}(x) \mid x \in A\} .$$

In other words, the height of (A, f) is the least non-negative integer $\lambda(f)$ satisfying $f^{\lambda(f)}(A) = f^{\lambda(f)+1}(A)$ and it is known as the *pre-period* of f [14, 15].

It is well-known that any algebra can be connected with monounary algebras by the notion of endomorphism [5, 6, 8, 12, 13]. For any finite algebra $\mathbf{A}$ and its endomorphism f, one can see that $|A| - 1$ is an upper bound of $\lambda(f)$. So there exists an endomorphism ψ, called *highest endomorphism*, with $\lambda(\psi) \geq \lambda(g)$ for all endomorphisms g of $\mathbf{A}$. In [1, 2, 3, 4], the authors focused on a finite lattice and showed that, for a finite modular $\mathbf{L}$, the pre-period of its endomorphism is less than or equal to the length of $\mathbf{L}$ where the *length* $\ell(\mathbf{L})$ of $\mathbf{L}$ is defined by $|C| - 1$ for the longest chain C in $\mathbf{L}$. A finite modular lattice $\mathbf{L}$ is said to have the *maximum pre-period property* (briefly MPP) if $\lambda(\mathbf{L}) = \ell(\mathbf{L})$. They gave a necessary and sufficient condition of the highest endomorphism of lattice having MPP.

A bounded distributive lattice $\mathbf{B}$ is said to be Boolean if, for each $a \in B$, there exists an element (unique) a', called the complement of a, such that $a \wedge a'$ is the bottom and $a \vee a'$ is the top. A Boolean lattice has MPP [3] and there are some highest endomorphisms. Consequently, it is interesting to find all highest endomorphisms of a Boolean lattice.

2 Preliminaries

Let $\mathbf{P}$ be an ordered set and let $x, y \in P$. We say that x is *covered by* y, written as $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$. An *n-element chain* is the ordered set $\{c_1 \prec c_2 \prec \ldots \prec c_n\}$, denoted by $\mathbf{n}$. It is well-known that a Boolean lattice is the direct power $\mathbf{2}^n$. A unary operation f of a lattice $\mathbf{L} = \langle L; \vee, \wedge \rangle$ is said to be an *endomorphism* if $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in L$.

The following result follows from Corollary 6 in [2].

Theorem 2.1. Let $\mathbf{L}$ be a finite modular lattice with the top 1, the bottom 0 and $\ell(\mathbf{L}) = n$ and let f be an endomorphism of $\mathbf{L}$. Then $\lambda(f) = n$ if and only if either

$$0 = f^n(1) \prec f^{n-1}(1) \prec \ldots \prec f(1) \prec 1$$

or

$$0 \prec f(0) \prec \ldots \prec f^{n-1}(0) \prec f^n(0) = 1.$$

Moreover, such f is highest.

Example 2.1. Let $\mathbf{2} = \{0 \prec 1\}$. The endomorphisms $f : \mathbf{2}^n \to \mathbf{2}^n$ and $f^\partial : \mathbf{2}^n \to \mathbf{2}^n$ defined by

$$f(a_1, \ldots, a_n) = (0, a_1, \ldots, a_{n-1})$$

and

$$f^\partial(a_1, \ldots, a_n) = (1, a_1, \ldots, a_{n-1})$$

for all $a_1, \ldots, a_n \in \{0, 1\}$ are highest.

3 All highest endomorphisms of $\mathbf{2}^n$

Let S_n be the set of all permutations on $\{1, \ldots, n\}$. For each set A, we consider an element $\bar{a} = (a_1, \ldots, a_n)$ in A^n as the function $\bar{a} : \{1, \ldots, n\} \to A$ defined by $\bar{a}(i) = a_i$ for all $i \in \{1, \ldots, n\}$. Note that $\mathbf{2}^n$ is a lattice having the length n with the top $\bar{1} = (1, \ldots, 1)$ and the bottom $\bar{0} = (0, \ldots, 0)$ and for each $\bar{a}, \bar{b} \in \mathbf{2}^n$,

$$(\bar{a} \vee \bar{b})(i) = \bar{a}(i) \vee \bar{b}(i)$$

and

$$(\bar{a} \wedge \bar{b})(i) = \bar{a}(i) \wedge \bar{b}(i)$$

for all $i \in \{1, \ldots, n\}$. We are going to define highest endomorphisms which are general forms of the functions in Example 2.1.

Theorem 3.1. For each $\sigma \in S_n$, define $\psi_\sigma : \mathbf{2}^n \to \mathbf{2}^n$ and $\psi_\sigma^\partial : \mathbf{2}^n \to \mathbf{2}^n$ by

$$\psi_\sigma(\bar{a})(\sigma(i)) = \begin{cases} \bar{a}(\sigma(i-1)) & \text{if } i > 1, \\ 0 & \text{if } i = 1 \end{cases}$$

and

$$\psi_\sigma^\partial(\bar{a})(\sigma(i)) = \begin{cases} \bar{a}(\sigma(i-1)) & \text{if } i > 1, \\ 1 & \text{if } i = 1 \end{cases}$$

for all $\bar{a} \in \mathbf{2}^n$. Then ψ_σ and ψ_σ^∂ are highest endomorphisms of $\mathbf{2}^n$ for all $\sigma \in S_n$.

Proof. Let $\sigma \in S_n$ and let $\bar{a}, \bar{b} \in \mathbf{2}^n$. Then for each $i \in \{1, \ldots, n\}$,

$$\begin{aligned}
\psi_\sigma(\bar{a} \vee \bar{b})(\sigma(i)) &= \begin{cases} (\bar{a} \vee \bar{b})(\sigma(i-1)) & \text{if } i > 1, \\ 0 & \text{if } i = 1 \end{cases} \\
&= \begin{cases} \bar{a}(\sigma(i-1)) \vee \bar{b}(\sigma(i-1)) & \text{if } i > 1, \\ 0 & \text{if } i = 1 \end{cases} \\
&= \psi_\sigma(\bar{a})(\sigma(i)) \vee \psi_\sigma(\bar{b})(\sigma(i)) \\
&= (\psi_\sigma(\bar{a}) \vee \psi_\sigma(\bar{b}))(\sigma(i)).
\end{aligned}$$

Hence, $\psi_\sigma(\bar{a} \vee \bar{b}) = \psi_\sigma(\bar{a}) \vee \psi_\sigma(\bar{b})$. Similarly, we get ψ_σ and ψ_σ^∂ are endomorphisms. Moreover, for each $1 \leq k \leq n$,

$$\psi_\sigma^k(\bar{1})(i) = \begin{cases} 1 & \text{if } i \notin \{\sigma(1), \sigma(2), \ldots, \sigma(k)\}, \\ 0 & \text{if } i \in \{\sigma(1), \sigma(2), \ldots, \sigma(k)\} \end{cases}$$

and

$$(\psi_\sigma^\partial)^k(\bar{0})(i) = \begin{cases} 1 & \text{if } i \in \{\sigma(1), \sigma(2), \ldots, \sigma(k)\}, \\ 0 & \text{if } i \notin \{\sigma(1), \sigma(2), \ldots, \sigma(k)\}. \end{cases}$$

These imply that

$$\bar{1} \succ \psi_\sigma(\bar{1}) \succ \cdots \succ \psi_\sigma^{n-1}(\bar{1}) \succ \psi_\sigma^n(\bar{1}) = \bar{0}$$

and

$$\bar{0} \prec \psi_\sigma^\partial(\bar{0}) \prec \cdots \prec (\psi_\sigma^\partial)^{n-1}(\bar{0}) \prec (\psi_\sigma^\partial)^n(\bar{0}) = \bar{1}.$$

Hence, ψ_σ and ψ_σ^∂ are highest. □

We see that ψ_ι and ψ_ι^∂ are the functions f and f^∂ in Example 2.1, respectively where ι is the identity map on $\{1, \ldots, n\}$.

Theorem 3.2. A highest endomorphism of $\mathbf{2}^n$ is exactly either ψ_σ or ψ_σ^∂ for some $\sigma \in S_n$.

Proof. Let f be a highest endomorphism of $\mathbf{2}^n$. Then $\lambda(f) = n$. Suppose that

$$\bar{1} \succ f(\bar{1}) \succ \cdots \succ f^{n-1}(\bar{1}) \succ f^n(\bar{1}) = \bar{0}.$$

Then there exists $j_1 \in \{1, \ldots, n\}$ such that

$$f(\bar{1})(i) = \begin{cases} 1 & \text{if } i \neq j_1, \\ 0 & \text{if } i = j_1. \end{cases}$$

Similarly, there exists $j_2 \in \{1, \dots, n\}$ such that

$$f^2(\bar{1})(i) = \begin{cases} 1 & \text{if } i \notin \{j_1, j_2\}, \\ 0 & \text{if } i \in \{j_1, j_2\}. \end{cases}$$

Proceeding in this manner, for each $1 \leq k \leq n$, there exists $j_k \in \{1, \dots, n\}$ such that

$$f^k(\bar{1})(i) = \begin{cases} 1 & \text{if } i \notin \{j_1, \dots, j_k\}, \\ 0 & \text{if } i \in \{j_1, \dots, j_k\}. \end{cases} \tag{3.1}$$

We define a permutation σ on $\{1, \dots, n\}$ by $\sigma(k) = j_k$ for all $k \in \{1, \dots, n\}$. We will show that $f = \psi_\sigma$. For each $j \in \{1, \dots, n\}$, the atom $\bar{a}_j$ of $\mathbf{2}^n$ is defined by

$$\bar{a}_j(i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It suffices to show that $f(\bar{a}_{\sigma(j)}) = \psi_\sigma(\bar{a}_{\sigma(j)})$ for all $j \in \{1, \dots, n\}$. Let $j \in \{1, \dots, n\}$ and, for convenience, let $\bar{a}_{\sigma(n+1)} = \bar{0}$. Since $\bar{a}_{\sigma(j+1)}(\sigma(1)) = 0 = \psi_\sigma(\bar{a}_{\sigma(j)})(\sigma(1))$ and for each $i \in \{2, \dots, n\}$

$$\begin{aligned} \psi_\sigma(\bar{a}_{\sigma(j)})(\sigma(i)) &= \bar{a}_{\sigma(j)}(\sigma(i-1)) \\ &= \begin{cases} 1 & \text{if } \sigma(i-1) = \sigma(j), \\ 0 & \text{if } \sigma(i-1) \neq \sigma(j) \end{cases} \\ &= \begin{cases} 1 & \text{if } \sigma(i) = \sigma(j+1), \\ 0 & \text{if } \sigma(i) \neq \sigma(j+1) \end{cases} \\ &= \bar{a}_{\sigma(j+1)}(\sigma(i)), \end{aligned}$$

we have $\psi_\sigma(\bar{a}_{\sigma(j)}) = \bar{a}_{\sigma(j+1)}$. By equation 3.1, we get

$$f(\bar{a}_{\sigma(m)} \vee \cdots \vee \bar{a}_{\sigma(n)}) = \bar{a}_{\sigma(m+1)} \vee \cdots \vee \bar{a}_{\sigma(n)} \tag{3.2}$$

for all $m \in \{1, \dots, n\}$. Assume that $f(\bar{a}_{\sigma(j)}) = (x_1, \dots, x_n)$. Then by equation 3.2,

$$\begin{aligned} \bar{a}_{\sigma(j+1)} \vee \cdots \vee \bar{a}_{\sigma(n)} &= f(\bar{a}_{\sigma(j)} \vee \cdots \vee \bar{a}_{\sigma(n)}) \\ &= f(\bar{a}_{\sigma(j)}) \vee f(\bar{a}_{\sigma(j+1)} \vee \cdots \vee \bar{a}_{\sigma(n)}) \\ &= (x_1, \dots, x_n) \vee \bar{a}_{\sigma(j+2)} \vee \cdots \vee \bar{a}_{\sigma(n)} \end{aligned}$$

which implies that $x_{\sigma(1)} = \cdots = x_{\sigma(j)} = 0$ and $x_{\sigma(j+1)} = 1$. Since

$$\begin{aligned}\bar{0} = f(\bar{0}) &= f(\bar{a}_{\sigma(j)} \wedge (\bar{a}_{\sigma(j+1)} \vee \cdots \vee \bar{a}_{\sigma(n)}))\\ &= f(\bar{a}_{\sigma(j)}) \wedge f(\bar{a}_{\sigma(j+1)} \vee \cdots \vee \bar{a}_{\sigma(n)})\\ &= (x_1, \ldots, x_n) \wedge \bar{a}_{\sigma(j+2)} \vee \cdots \vee \bar{a}_{\sigma(n)},\end{aligned}$$

we get $x_{\sigma(j+2)} = \cdots = x_{\sigma(n)} = 0$. Hence $f(\bar{a}_{\sigma(j)}) = \bar{a}_{\sigma(j+1)} = \psi_\sigma(\bar{a}_{\sigma(j)})$. Since j is arbitrary and σ is a permutation on $\{1, \ldots, n\}$, we have $f = \psi_\sigma$. Similarly, if

$$\bar{0} \prec f(\bar{0}) \prec \cdots \prec f^{n-1}(\bar{0}) \prec f^n(\bar{0}) = \bar{1},$$

then $f = \psi_\sigma^\partial$ for some $\sigma \in S_n$. □

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