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On the Exponential Diophantine equation $5^x - 3^y = z^2$

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Abstract

In this work, we show that $(0, 0, 0), (1, 0, 2)$, and $(2, 2, 4)$ are all the solutions of the exponential Diophantine equation $5^x - 3^y = z^2$, where x, y, z are non-negative integers.

1 Introduction

For over two decades, Exponential Diophantine Equations have been widespread problems in Number Theory. In 2004, Mihailescu [4] proved Catalan's conjecture that the exponential Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$, has only one solution $(a, b, x, y) = (3, 2, 2, 3)$. This settled conjecture has been used in finding integer solutions of many Exponential Diophantine Equations. In 2007, Acu [1] proved that $2^x + 5^y = z^2$ has exactly the two solutions $(3, 0, 3), (2, 1, 3)$ in non-negative integers. In (2011), Suvarnamani et al. [6] studied the two equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. In 2018, Rabago [5] discovered all solutions of the Diophantine Equation $4^x - p^y = z^2$. Moreover, he discovered all solutions of $4^x - p^y = 3z^2$, where p is a prime and $p \equiv 3 \mod 4$. In

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2019, Thongnak et al. [7] studied the equation $2^x - 3^y = z^2$ by applying Mihailescu's result to prove that there are three solutions to the equation. In the same year, Burshtein [3] suggested that the Exponential Diophantine Equation $6^x - 11^y = z^2$ has no positive integer solutions when $2 < x \leq 16$. In 2020, Buosi et al. [2] discovered all positive solutions of the Diophantine Equation $p^x - 2^y = z^2$ with $p = k^2 + 2$ where p is a prime number and $k \geqslant 0$.

Although many of the Exponential Diophantine Equations have been studied, there still remain many unsolved problems. In this work, we find the non-negative integer solutions of the Exponential Diophantine Equation $5^x - 3^y = z^2$.

2 Preliminaries

In this part, the basic knowledge of number theory is given to compute and prove all the non-negative integer solutions to the equation.

Definition 2.1. *If n is a positive integer and $\gcd(a, n) = 1$, the least positive integer k such that $a^k \equiv 1 \mod n$ is called the order of a modulo n and is denoted by $\mathrm{ord}_n a$.*

Theorem 2.2. *Let the integer a have order k modulo n. Then $a^h \equiv 1 \mod n$ if and only if $k|h$; in particular, $k|\phi(n)$.*

Theorem 2.3. *(Euclid's Lemma) If $a|bc$ and $(a, b) = 1$, then $a|c$.*

Lemma 2.4. *(Catalan's conjecture) [4] Let a, b, x and y be integers. The Diophantine equation $a^x - b^y = z^2$ with $\min\{a, b, x, y\} > 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$.*

Theorem 2.5. *If $a|c$, $b|c$ and $(a, b) = 1$, then $ab|c$.*

3 Main results

Theorem 3.1. *Let x, y, and z be non-negative integers. The Diophantine equation $5^x - 3^y = z^2$ has the three solutions, $(x, y, z) = (0, 0, 0), (1, 0, 2)$, and $(2, 2, 4)$.*

Proof. Let x, y, and z be non-negative integers such that

$$5^x - 3^y = z^2. \tag{3.1}$$

We begin the proof by considering the following four cases:
Case 1: $x = 0, y = 0$. From (3.1), we obtain $z^2 = 0$ or $z = 0$. Hence $(x, y, z) = (0, 0, 0)$ is a solution.
Case 2: $x = 0, y > 0$. From (3.1), we have $z^2 = 1 - 3^y < 0$, which is impossible.
Case 3: $x > 0, y = 0$. (3.1) becomes

$$5^x - z^2 = 1. \tag{3.2}$$

If $x = 1$, then $z^2 = 4$ or $z = 2$. Thus $(x, y, z) = (1, 0, 2)$ is a solution.
If $x > 1$, then 3.2 yields $z > 1$. By Lemma 2.4 (Catalan's conjecture), we can see that (3.2) has no solution for $x > 1$.
Case 4: $x > 0, y > 0$. Equation (3.1) implies that $z^2 \equiv (-1)^x \mod 3$ but z^2 is not equivalent to $-1 \mod 3$. Thus x must be even. Let $x = 2k$, $\exists k \in \mathbb{Z}^+$. From (3.1), we obtain $3^y = 5^{2k} - z^2 = (5^k - z)(5^k + z)$. There exists $\alpha \in \mathbb{Z}^+ \cup \{0\}$ such that $5^k - z = 3^\alpha$ and $5^k + z = 3^{y-\alpha}$, where $\alpha < y - \alpha$. We have $2 \cdot 5^k = 3^{y-\alpha} + 3^\alpha = 3^\alpha(3^{y-2\alpha} + 1)$. Since $3 \nmid 2 \cdot 5^k$, $\alpha = 0$ and

$$2 \cdot 5^k = 3^y + 1. \tag{3.3}$$

We consider y as follows:
If $y = 1$, then (3.3) becomes $2 \cdot 5^k = 4$. Thus $5^k = 2$, which is impossible.
If $y = 2$, then (3.3) becomes $2 \cdot 5^k = 10$. We obtain $k = 1$ and so $x = 2$ and $z = 4$. Hence the solution of (3.1) is $(2, 2, 4)$.
If $y > 2$, then (3.3) becomes $k > 1$ and $2 \cdot 5^k - 10 = 3^y - 9$ or $10(5^{k-1} - 1) = 9(3^{y-2} - 1)$. Let $m = k - 1 > 0$ and $n = y - 2 > 0$. We obtain

$$10(5^m - 1) = 9(3^n - 1). \tag{3.4}$$

From (3.4), $5 | 9(3^n - 1)$. Since $\gcd(5, 9) = 1$, we also obtain $3^n \equiv 1 \mod 5$. Since $\text{ord}_5 3 = 4$, $4 | n$. Again by (3.4), we find that $9 | 10(5^m - 1)$. This means that $9 | 5^m - 1$ or $5^m \equiv 1 \mod 9$ because $\gcd(9, 10) = 1$. Since $\text{ord}_9 5 = 6$, $5^m \equiv 1 \mod 9$ implies that $6 | m$. That is, $m = 6t$, $\exists\, t \in \mathbb{Z}^+$. By considering (3.4), since $5^{6t} \equiv 1 \mod 31$, we then obtain $31 | 9(3^n - 1)$. With $\gcd(9, 31) = 1$, this implies that $31 | 3^n - 1$ or $3^n \equiv 1 \mod 31$. Since $\text{ord}_{31} 3 = 30$, we obtain $30 | n$ which implies that $5 | n$. Now, $4 | n$ and $5 | n$ with $\gcd(4, 5) = 1$. So $20 | n$. Assume $n = 20l$, $\exists\, l \in \mathbb{Z}^+$. We have $3^n = 3^{20l} \equiv 1 \mod 25$ or $25 | 3^n - 1$. Again by (3.4), we obtain $25 | 10(5^m - 1)$ or $5 | 2(5^m - 1)$. Since $\gcd(2, 5) = 1$, we can write $5 | 5^m - 1$, which is impossible. The proof is now complete. $\square$

4 Conclusion

In this work, we have found all the non-negative integer solutions of the exponential Diophantine Equation $5^x - 3^y = z^2$ using four cases based on the x and y values. The non-negative integer solution set is $\{(0,0,0),(1,0,2),(2,2,4)\}$.

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