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Pseudo NQ-principally Projective Modules

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Abstract

Let R be an associative ring with identity. Let M be a right R-module. A right R-module N is called *pseudo nonessential M-principally projective* (briefly, pseudo *NM-principally projective*) if, for each $s \in S$ with $s(M) \not\subset^e M$, any R-epimorphism from N to $s(M)$ can be lifted to an R-homomorphism from N to M. M is called *pseudo nonessential quasi-principally projective* (briefly, pseudo *NQ-principally projective*) if, it is pseudo *NM-principally projective*. In this paper, we give some characterizations and properties of pseudo NQ-principally projective modules.

1 Introduction

Throughout this paper, R will be an associative ring with identity and all modules are unitary right R-modules. For right R-modules M and N, $Hom_R(M, N)$ denotes the set of all R-homomorphisms from M to N and $S = End_R(M)$ denotes the endomorphism ring of M. If X is a subset of M,

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the right (resp. left) annihilator of X in R (resp.S) is denoted by $r_R(X)$ (resp. $l_S(X)$). A pair (E, ι) is an *injective envelope* of M in case E is an injective R-module and $0 \to M \xrightarrow{\iota} E$ is an essential R-monomorphism. The injective envelope of M is denoted by $E(M)$. By the notation $N \subset^{\oplus} M (N \subset^{e} M)$ we mean that N is a direct summand (an essential submodule) of M.

Let R be a ring. A right R-module M is called *principally injective* (or P-injective), if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$ where l and r are left and right annihilators, respectively. In [4], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. Nicholson, Park, and Yousif [5] extended this notion of principally injective rings to the one for modules.

Sanh et al. [6] extended this notion to modules. A right R-module N is called *M-principally injective* if every R-homomorphism from an M-cyclic submodule of M to N can be extended to an R-homomorphism from M to N. Tansee and Wongwai [7] introduced the dual notion, a right R-module N is called *M-principally projective* if every R-homomorphism from N to an M-cyclic submodule of M can be lifted to an R-homomorphism from N to M. M is called *quasi-principally* (or *semi-*) *projective* if it is M-principally projective. In this note we introduce the definition of pseudo NQ-principally projective modules and give some characterizations and properties.

2 Pseudo NM-principally Projective Modules

Recall that a submodule K of a right R module M is *essential* (or *large*) in M if, for every nonzero submodule L of M, we have $K \cap L \neq 0$.

Definition 2.1. *Let M be a right R-module. A right R-module N is called pseudo nonessential M-principally projective (briefly, pseudo NM-principally projective) if, for each $s \in S$ with $s(M) \not\subset^{e} M$, any R-epimorphism from N to $s(M)$ can be lifted to an R-homomorphism from N to M.*

Lemma 2.2. *(1) Any direct summand of pseudo NM-principally projective module is again pseudo NM-principally projective.*
(2) If $s \in S$ with $s(M) \not\subset^{e} M$, and $s(M)$ is pseudo NM-principally projective, then $Ker(s) \subset^{\oplus} M$ and $s(M) \simeq K \subset^{\oplus} M$.

Proof. (1) By definition.
(2) Let $s \in S$ with $s(M) \not\subset^{e} M$. Then there exists an R-homomorphism

$\varphi : s(M) \to M$ such that $s\varphi = 1_{s(M)}$. Then by [1, Lemma 5.1], s is a split R-epimorphism. There $M = Ker(s) \oplus K$, where $s(M) \simeq K$. □

Example 2.3. *(1) If N is pseudo NM-principally projective and $N \simeq X$, then X is pseudo NM-principally projective.*
(2) Every M-principally projective module is pseudo NM-principally projective.
(3) Let $\mathbb{Z}$ be the set of integers. Then the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is pseudo nonessential $\mathbb{Z}/4\mathbb{Z}$-principally projective, but not $\mathbb{Z}/4\mathbb{Z}$-projective.

Proposition 2.4. *Let M be a right R-module. Then N is pseudo NM-principally projective if and only if N is pseudo NK-principally projective for every M-cyclic submodule K of M.*

Proof. ($\Rightarrow$) Write $K = s(M)$. Let $g \in End_R(K)$ with $g(K) \not\subset^e K$ and let $\varphi : N \to g(K)$ be an R-epimorphism. Since $gs(M) \not\subset^e M$, φ can be lifted to an R-homomorphism $\hat{\varphi} : N \to M$. Hence $s\hat{\varphi}$ lifts φ. Therefore N is pseudo NK-principally projective.
($\Leftarrow$) is clear. □

Proposition 2.5. *Let M and N be right R-modules.*
Then the following are equivalent:
(1) N is pseudo NM-principally projective.
(2) For each $s \in S$ with $s(M) \not\subset^e M$,

$$\{\varphi \in Hom_R(N, M) | \varphi(N) = s(M)\} \subset sHom_R(N, M).$$

(3) For each $s \in S$ with $s(M) \not\subset^e M$,

$$\{\varphi \in Hom_R(N, M) | \varphi(N) = s(M)\} = s\{\varphi \in Hom_R(N, M) | \varphi(N) + Ker(s) = M\}.$$

Proof. (1) $\Rightarrow$ (2) Let $s \in S$ with $s(M) \not\subset^e M$ and let $\varphi \in Hom_R(N, M)$ such that $\varphi(N) = s(M)$. Since N is pseudo NM-principally projective, there exists an R-homomorphism $\hat{\varphi} : N \to M$ such that $\varphi = s\hat{\varphi}$. It follows that $\varphi \in sHom_R(N, M)$.
(2) $\Rightarrow$ (3) It is clear that $s\{\varphi \in Hom_R(N, M) | \varphi(N) + Ker(s) = M\} \subset \{\varphi \in Hom_R(N, M) | \varphi(N) = s(M)\}$. Conversely, let $\alpha \in Hom_R(N, M)$ such that $\alpha(N) = s(M)$. Then by (2) we have $\alpha \in sHom_R(N, M)$, so $\alpha = s\hat{\varphi}$ for some $\hat{\varphi} \in Hom_R(N, M)$. Then $\alpha = s\hat{\varphi} \in s\{\varphi \in Hom_R(N, M) | \varphi(N) + Ker(s) = M\}$
(3) $\Rightarrow$ (1) Let $s \in S$ with $s(M) \not\subset^e M$ and let $\varphi : N \to s(M)$ be an R-epimorphism. Then $\varphi(N) = s(M)$ and hence by (3) we have $\varphi = s\hat{\varphi}$ for

some R-homomorphism $\hat{\varphi} : N \to M$ with $\hat{\varphi}(N) + Ker(s) = M$. Then N is pseudo NM-principally projective. □

Corollary 2.6. *Let M be an injective module.*
If every nonessential M-cyclic submodule of M is injective, then every submodule of pseudo NM-principally projective is pseudo NM-principally projective.

Proof. Clear. □

3 Pseudo NQ-principally Projective Modules

A right R-module M is called pseudo nonessential quasi-principally projective (briefly, pseudo NQ-principally projective) if it is pseudo NM-principally projective. It is clear that, any direct summand of a pseudo NQ-principally projective module is again pseudo NQ-principally projective.

Proposition 3.1. *Let M be a right R-modules.*
Then the following are equivalent:
(1) M is pseudo NQ-principally projective.
(2) For each $s, t \in S$ with $t(M) \not\subset^e M$, if $t(M) = s(M)$ then $sS = tS$.
(3) For each $s, t \in S$ with $ts(M) \not\subset^e M$,

$$\{f \in S | tf(M) = ts(M)\} \subset sS + \{v \in S | v(M) \subset Ker(t)\}.$$

Proof. (1) ⇒ (2) Let $s, t \in S$ with $t(M) \not\subset^e M$ and $t(M) = s(M)$. Then by (1), s can be lifted to an R-homomorphism $\hat{\varphi} \in S$. Hence $s = t\hat{\varphi} \in tS$, so $sS \subset tS$. The same argument shows that $tS \subset sS$.
(2) ⇒ (3) Let $g \in S$ such that $tg(M) = ts(M)$. Since $ts(M) \not\subset^e M$, by (2) we have $tgS = tsS$. Then $tg \in tsS$ so $tg = tsf$ for some $f \in S$. It follows that $g - sf = h$ for some $h \in S$ with $h(M) \subset Ker(t)$. Hence $g = sf + h \in sS + \{v \in S | v(M) \subset Ker(t)\}$.
(3) ⇒ (1) Let $s \in S$ with $s(M) \not\subset^e M$ and let $\varphi : M \to s(M)$ be an R-epimorphism. Then $\varphi(M) = s(M)$ and hence by (3) and put $t = 1$, $\{f \in S | f(M) = s(M)\} \subset sS + \{v \in S | v(M) \subset Ker(1)\} = sS$. Hence $\varphi \in sS$ so $\varphi = s\hat{\varphi}$ for some $\hat{\varphi} \in S$. Then N is pseudo NM-principally projective. □

Lemma 3.2. *Let P be a projective module and $P \oplus K$ is pseudo NQ-principally projective. If there is an R-epimorphism $g : P \to K$, then K is projective.*

Proof. Let $\pi_1 : P \oplus K \to P$ be the projection map. Since $P \oplus K$ is pseudo NQ-principally projective and $g\pi_1(P \oplus K) \not\subset^e P \oplus K$, there exists an R-homomorphism $\beta : P \oplus K \to P \oplus K$ such that $g\pi_1\beta = \pi_2$ where $\pi_2 : P \oplus K \to K$ is the projection map. Then $1_k = \pi_2\iota_2 = g\pi_1\beta\iota_2$ where $\iota_2 : K \to P \oplus K$ is the injective map. Put $\hat{\varphi} = \pi_1\beta\iota_2$, so $1_k = g\hat{\varphi}$. Then by [1, Lemma 5.1], g is a split R-epimorphism. Hence there exists a submodule X of P such that $X \simeq K$ and $P = Ker(f) \oplus X$. Therefore K is projective. □

Lemma 3.3. *Let E be an injective module and $E \oplus N$ is quasi-principally injective. If there is an R-monomorphism $\varphi : N \to E$, then N is injective.*

Proof. Since N is an $E \oplus N$-cyclic submodule of $E \oplus N$, there exists an R-homomorphism $\alpha : E \oplus N \to E \oplus N$ such that $\alpha\iota_1\varphi = \iota_2$ where $\iota_1 : E \to E \oplus N$ and $\iota_2 : N \to E \oplus N$ are the injection maps. Then $\pi_2\alpha\iota_1\varphi = \pi_2\iota_2 = 1_N$ where $\pi_2 : E \oplus N \to N$ is the projection map. Hence the R-monomorphism φ splits. It follows that $E = \varphi(N) \oplus D$ for some a submodule D of E. Then $\varphi(N)$ is injective and hence N is injective. □

A ring R is right hereditary [1] in case of its right ideals is projective. Equivalently, every submodule of a projective

Proposition 3.4. *The following conditions are equivalent for a ring R.*
(1) R is right hereditary.
(2) Every submodule of a projective R-module is pseudo NQ-principally projective.
(3) Every factor module of an injective R-module is quasi-principally injective.
(4) Every sum of two injective submodules of an R-module is quasi-principally injective.
(5) Every sum of two isomorphic injective submodules of an R-module is quasi-principally injective.

Proof. $(1) \Rightarrow (2), (1) \Rightarrow (3)$ and $(4) \Rightarrow (5)$ are clear.
$(2) \Rightarrow (1)$ Let P be a projective R-module and let K be a submodule of P. We must show that K is projective. Let $\varphi : F \to K$ be an R-epimorphism, where F is a free module. Then $F \oplus K$ is a submodule of a projective R-module $F \oplus P$. Then by (2), $F \oplus K$ is pseudo NQ-principally projective. Hence K is projective by Lemma 3.2. Therefore R is right hereditary.
$(3) \Rightarrow (1)$ Let E be an injective R-module, let N be a submodule of E, and let $\eta : E \to E/N$ be the natural R-epimorphism. Then we have an R-epimorphism:

$$\iota + \eta : E(E/N) \oplus E \to E(E/N) \oplus E/N.$$

It follows that $(E(E/N) \oplus E)/Ker(\iota + \eta) \simeq E(E/N) \oplus E/N$. Then by (3), $(E(E/N) \oplus E)/Ker(\iota + \eta)$ is quasi-principally injective. Hence $E(E/N) \oplus E/N$ is quasi-principally injective and we have an R-monomorphism, $E/N \to E(E/N)$ so E/N is injective by Lemma 3.3. Therefore R is right hereditary.
(3) $\Rightarrow$ (4) Let E_1 and E_2 be two injective submodules of an R-module M. Since $E_1 \oplus E_2$ is injective and there exists an R-epimorphism $\alpha : E_1 \oplus E_2 \to E_1 + E_2$, then $(E_1 \oplus E_2)/Ker(\alpha)$ is quasi-principally injective by (3). Since $(E_1 \oplus E_2)/Ker(\alpha) \simeq E_1 + E_2$, $E_1 + E_2$ is quasi-principally injective.
(5) $\Rightarrow$ (3) By the similar proof to (6) $\Rightarrow$ (4) of Theorem 4 in [9]. □

A right R-module M is called a *duo module* if every submodule of M is fully invariant. M satisfies (D_2) [3] if, A is a submodule of M such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M, M satisfies (D_3) if, M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$ then $M_1 \cap M_2$ is a direct summand of M. The next lemma shows that conditions (D_2) and (D_3) also hold in pseudo NQ principally projective.

Lemma 3.5. *If M is a pseudo NQ-principally projective module, then M satisfies (D_2) and (D_3)*

Proof. (D_2) Let B be a direct summand of M, A a submodule of M and let $\varphi : M/A \to B$ be an R-isomorphism. Define $\alpha : M \to B$ by $\alpha(m) = \alpha\eta(m)$ for every $m \in M$ and $\eta : M \to M/A$ is the natural R-epimorphism. It is clear that α is an R-epimorphism and $Ker(\alpha) = A$. Since B is a direct summand of M and M is pseudo NQ-principally projective, B is pseudo NM-principally projective by Lemma 2.2. We have B is a nonessential M-cyclic submodule of M, then there exists an R-homomorphism $\beta : B \to M$ such that $\alpha\beta = 1_B$. Then α is a split R-epimorphism. It follows that $M = Ker(\alpha) \oplus K$ for some a submodule K of M. Then A is a direct summand of M.

(D_3) Let A and B are direct summand of M with $A + B = M$. Write $M = A \oplus A'$ where A' is a submodule of M. Since $A' \simeq (A + B)/A$ and $(A + B)/A \simeq B/(A \cap B)$, $A \cap B$ is a direct summand of M by (D_2) □

Lemma 3.6. *If M is duo pseudo NQ-principally projective and $s \in S$ with $M = s(M) \oplus X$, then $X = Ker(s)$.*

Proof. Since M is duo, $s(X) \subset s(M) \cap X = 0$, so $X \subset Ker(s)$. Now we have $M = s(M) + Ker(s)$ and $M/Ker(s) \simeq s(M)$. Then $Ker(s) \subset^{\oplus} M$ by (D_2). It follows that $s(M) \cap Ker(s) \subset^{\oplus} M$ by (D_3). Write $M = (s(M) \cap Ker(s)) \oplus N$. Since M is duo, $s(M) = s(N) \subset s(M) \cap N$ so $s(M) \subset N$. It follows that $s(M) \cap Ker(s) = 0$. Therefore $M = s(M) \oplus Ker(s)$, and $X = Ker(s)$. □

M is said [8] to have the summand intersection property (SIP) if the intersection of two direct summands is again a direct summand. The module M is said [2] to have the summand sum property (SSP) if the sum of any two summands of M is again a summand.

A right R-module M satisfies (C_2) [3] if, a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M. M satisfies (C_3) if, M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is a direct summand of M. It is clear that if, M satisfies (C_2) then it satisfies (C_3).

Proposition 3.7. *Let M be a pseudo NQ-principally projective module. If M is a quasi-principally injective and $s \in S$ with $s(M) \not\subset^e M$, then the following statements are equivalent:*
(1) $s(M)$ is a direct summand of M.
(2) $s(M)$ is pseudo NM-principally projective.
(3) $s(M)$ is M-principally injective.

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (3) Since $s(M) \not\subset^e M$ and by (2), there exists an R-homomorphism $\alpha : s(M) \to M$ such that $s\alpha = 1_{s(M)}$ so s splits. Then $M = Ker(s) \oplus D$ for some submodule D of and $s(M) \simeq D$. Then $s(M)$ is a direct summand of M by (C_2) hence $s(M)$ is M-principally injective.
(3) $\Rightarrow$ (1) Since $s(M)$ is M-principally injective, $\iota\alpha = 1_{s(M)}$ for some an R-homomorphism $\alpha : M \to s(M)$ and $\iota : s(M) \to M$ is the inclusion map. It follows that $s(M) \subset^{\oplus} M$. □

Proposition 3.8. *Let M be a duo and pseudo NQ-principally projective module. Then*
(1) M has the (SIP).
(2) In addition, if M has the property (C_2), then M has the (SSP).

Proof. (1) Write $M = s(M) \oplus K$ and $M = t(M) \oplus L$. Since M is duo, $s(M) = s(t(M) \oplus L) = s(t(M)) + s(L) \subset (s(M) \cap t(M)) \oplus (s(M) \cap L) \subset s(M)$. Then $s(M) \cap t(M) \subset^{\oplus} M$.
(2) From (1), we write $M = (s(M) \cap t(M)) \oplus N$. Then $t(M) = t(M) \cap ((s(M) \cap t(M)) \oplus N) = t(M) \cap s(M) \oplus t(M) \cap N$ by the Modular law. Hence $s(M) + t(M) = s(M) + (s(M) \cap t(M) \oplus t(M) \cap N) = s(M) \oplus t(M) \cap N$. Since $M = (s(M) \cap t(M)) \oplus N \subset t(M) + N \subset M, t(M) + N = M$. Then $t(M) \cap N \subset^{\oplus} M$ by (D_3). Therefore $s(M) + t(M) \subset^{\oplus} M$. □

Theorem 3.9. *Let M be a pseudo NQ-principally projective module. Then S is regular if and only if for each $s \in S$, there exists an idempotent $\alpha \in S$ such that $s(M) = \alpha(M)$.*

Proof. ($\Rightarrow$) Clear.
($\Leftarrow$) Let $s \in S$. Then $s(M) = \alpha(M)$ where $\alpha \in S$ is an idempotent. Since $s(M) \subset^{\oplus} M$, $sS = \alpha S$ by Proposition 3.1. Therefore $sS \subset^{\oplus} S$. $\square$

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References

[1] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, 1992.

[2] J. L. Garcia, Properties of direct summands of modules, Comm. Algebra, **17,** no. 1, (1989), 73-92.

[3] S. H. Mohamed, B. J. Muller, Continuous and Discrete Modules, 1990.

[4] W. K. Nicholson, M. F. Yousif, Principally Injective Rings, J. Algebra, **174,** (1995), 77–93.

[5] W. K. Nicholson, J. K. Park, M. F. Yousif, Principally quasi-injective modules, Comm. Algebra, **27,** no. 4, (1999), 1683-1693.

[6] N. V. Sanh, K. P. Shum, S. Dhompongsa, S. Wongwai, On quasi-principally injective modules, Algebra Coll., **6,** no. 3, (1999), 269–276.

[7] H. Tansee, S. Wongwai, A Note on Semi-Projective Modules, Kyungpook Math. J., **42,** (2002), 369–380.

[8] G. V. Wilson, Modules with the summand intersection property, Comm. Algebra, **14,** (1986), 21–38.

[9] Weimin Xue, Characterization of rings using direct-projective modules and direct-injective modules, **87,** (1993), 99–104.