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Hadamard Determinant Inequalities for Accretive-Dissipative Matrices

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Abstract

In this article, we present new bounds for determinant inequalities involving accretive-dissipative matrices.

1 Introduction

Let $M_n(\mathbb{C})$ be the algebra of all $n\times n$ complex matrices. A matrix $S \in M_n(\mathbb{C})$ is called accretive-dissipative if in its Cartesian decomposition $S = A + iB$, the matrices A and B are positive definite, where $A = Re(S) = \frac{S+S^*}{2}$ and $B = Im(S) = \frac{S-S^*}{2i}$.
In this paper, we present the accretive-dissipative version for determinant inequalities and we get some new other bounds. Hadamard inequality states

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that if $A = [a_{ij}] \in M_n(\mathbb{C})$ is positive definite, then

$$detA \leq \prod_{i=1}^{n} a_{ii}. \tag{1.1}$$

Also, we give some results for accretive-dissipative matrices using the Hadamard product.
Let $A = [a_{ij}] \in M_n(\mathbb{C})$ and $B = [b_{ij}] \in M_n(\mathbb{C})$. Then the Hadamard product (Schur product) of A and B is $A \circ B$ which is defined as $A \circ B = [a_{ij}b_{ij}] \in M_n(\mathbb{C})$.
Like usual matrix product, Hadamard product distributes over matrix addition. That is, for $A, B, C \in M_n(\mathbb{C})$, we have $A \circ (B + C) = A \circ B + A \circ C$. Moreover, $A \circ (k)B = k(A \circ B) = (kA) \circ B$ for $k \in \mathbb{C}$.
It is known that if A and B are positive definite matrices, then $A \circ B$ is positive definite which implies that $det(A \circ B) \geq 0$. Also, it is known that, if $A, B \in M_n(\mathbb{C})$ such that $A \geq 0$ and $B \geq 0$, then

$$det(A \circ B) \geq (detA)(detB). \tag{1.2}$$

In this section, we present some lemmas that are needed in the proof of our main results.

Lemma 1.1. *[5] Let $B, C \in M_n(\mathbb{C})$ be positive semidefinite. Then*

$$|det(B + iC)| \leq det(B + C) \leq 2^{\frac{n}{2}}|det(B + iC)|. \tag{1.3}$$

Lemma 1.2. *[4] Let $A \in M_n(\mathbb{C})$ be such that $A = H + iK$, where H is positive semidefinite and K is Hermitian, then*

$$|detA| = |det(H + iK)| \geq |detK| + |detH|. \tag{1.4}$$

Lemma 1.3. *[1] If $A, B \in M_n(\mathbb{C})$ are positive definite, then*

$$det(I + A + B) \leq det(I + A)det(I + B). \tag{1.5}$$

Lemma 1.4. *[6] Let A,B,C be positive semidefinite matrices. Then*

$$det(A + B + C) + detC \geq det(A + C) + det(B + C). \tag{1.6}$$

2 Main results

Theorem 2.1. *Let $T = [t_{ij}] \in M_n(\mathbb{C})$ be accretive-dissipative. Then*

$$|detT| \leq 2^{\frac{n}{2}} \prod_{j=1}^{n} |t_{jj}|.$$

Proof.
Let $T = A + iB$ be the Cartesian decomposition of T. Then

$$\begin{aligned} |detT| &= |det(A + iB)| \\ &\leq det(A + B) \qquad (By \quad Lemma 1.1) \\ &\leq \prod_{j=1}^{n} (a_{jj} + b_{jj}). \qquad (By \quad Inequality 1.1) \end{aligned}$$

Now, notice that for any positive numbers a_j and b_j, we have $a_j + b_j \leq \sqrt{2}|a_j + ib_j|$, for $j = 1, 2, 3, \cdots, n$. So

$$|detT| \leq \prod_{j=1}^{n} \sqrt{2}|a_{jj} + ib_{jj}|$$

$$= 2^{\frac{n}{2}} \prod_{j=1}^{n} |t_{jj}|.$$

Remark 2.2. *It can be seen that the determinant inequality in Theorem 2.1 is sharp. This can be demonstrated by considering the accretive-dissipative matrix $T = \begin{pmatrix} 1+i & -1+i \\ -1+i & 1+i \end{pmatrix}$. Note that*

$$|detT| = (detT^*T)^{1/2} = 4 = 2 \prod_{j=1}^{n} |t_{jj}|.$$

Theorem 2.3. *If $S, T \in M_n(\mathbb{C})$ are accretive-dissipative, then*

$$|det(I + S + T)| \leq 2^n |det(I + S)||det(I + T)|.$$

Proof.
Let $S = A + iB$ and $T = C + iD$ be the Cartesian decompositions of S and T, respectively. Then

$$\begin{aligned}|det(I+S+T)| &= |det(I+A+iB+C+iD)| \\ &= |det((I+A+C)+i(B+D))| \\ &\leq |det((I+A+C)+(B+D))| \qquad (By \quad Lemma 1.1) \\ &= |det(I+(A+B)+(C+D))| \\ &\leq det(I+A+B)det(I+C+D) \qquad (By \quad Lemma 1.3) \\ &\leq 2^{n/2}|det(I+A+iB)|2^{n/2}|det(I+C+iD)| \qquad (By \quad Lemma 1.1) \\ &= 2^n|det(I+S)||det(I+T)|.\end{aligned}$$

Corollary 2.4. *Let $S \in M_n(\mathbb{C})$ be accretive-dissipative and let $S = A + iB$ be the Cartesian decomposition of S. Then*

$$det(I + S^*S + SS^*) \leq (det(I + A^2))^2(det(I + B^2))^2.$$

Proof.

$$\begin{aligned}det(I+S^*S+SS^*) &= det(I+2A^2+2B^2) \\ &\leq det(I+2A^2)det(I+2B^2) \qquad (By \quad Lemma 1.3) \\ &\leq det(I+A^2)det(I+A^2)det(I+B^2)det(I+B^2)\end{aligned}$$

Thus, the result is obvious.

Now, we present an attractive extension for Lemma 1.4 that can be found in [[6], p.215, Problem 36] using accretive-dissipative matrices.

Theorem 2.5. *If S, T and $W \in M_n(\mathbb{C})$ are accretive-dissipative, then*

$$|det(S+T+W)| + |detW| \geq 2^{\frac{-n}{2}}(|det(S+W)| + |det(T+W)|).$$

Proof.
Let $S = A+iB, T = C+iD$ and $W = E+iF$ be the Cartesian decompositions of S, T and W respectively. Then

$$|det(S+T+W)|+|detW| = |det((A+iB)+(C+iD)+(E+iF))|+|det(E+iF)|$$

$$\geq 2^{\frac{-n}{2}} det((A+C+E)+(B+D+F)) + 2^{\frac{-n}{2}} det(E+F). \qquad (By \quad Lemma 1.1)$$
$$= 2^{\frac{-n}{2}} (det((A+B)+(C+D)+(E+F)) + det(E+F))$$
$$\geq 2^{\frac{-n}{2}} (det(A+B+E+F) + det(C+D+E+F)) \qquad (By \quad Lemma 1.4)$$
$$= 2^{\frac{-n}{2}} (det((A+E)+(B+F)) + det((C+E)+(D+F)))$$
$$\geq 2^{\frac{-n}{2}} (|det((A+E)+i(B+F))| + |det((C+E)+i(D+F))|)$$
$$= 2^{\frac{-n}{2}} (|det(S+W)| + |det(T+W)|).$$

Theorem 2.6. *Let $S \in M_n(\mathbb{C})$ be accretive-dissipative where $S = A + iB$ is its Cartesian decomposition and let $C \in M_n(\mathbb{C})$ be positive definite. Then*

$$|det(S \circ C)| \geq detC(detA + detB).$$

Proof.
We have

$$\begin{aligned}
|det(S \circ C)| &= |det((A+iB) \circ C)| \\
&= |det((A \circ C) + i(B \circ C))| \\
&\geq det(A \circ C) + det(B \circ C) \qquad (By \quad Lemma 1.2) \\
&\geq detA detC + detB detC \qquad (By \quad Inequality 1.2) \\
&= detC(detA + detB).
\end{aligned}$$

Theorem 2.7. *Let $S \in M_n(\mathbb{C})$ be accretive-dissipative where $S = A + iB$ is its Cartesian decomposition and let $C \in M_n(\mathbb{C})$ be positive definite. Then*

$$|det(S \circ C)| \geq 2^{-n/2} detC(detA + detB).$$

Proof.
We have

$$\begin{aligned}
|det(S \circ C)| &= |det((A \circ C) + i(B \circ C))| \\
&\geq 2^{-n/2} det((A \circ C) + (B \circ C)) \qquad (By \quad Lemma 1.1) \\
&\geq 2^{-n/2} (det(A \circ C) + det(B \circ C)) \qquad (By \quad Lemma 1.2) \\
&\geq 2^{-n/2} (detA detC + detB detC) \qquad (By \quad Inequality 1.2) \\
&= 2^{-n/2} detC(detA + detB).
\end{aligned}$$

It should be mentioned that the bound for $|det(S \circ C)|$ in Theorem 2.7 is better than the bound in Theorem 2.6.

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