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Bi-Periodic k-Pell Sequence

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Abstract

In this paper, we define the bi-periodic k-Pell sequence. We obtain Binet's formula, some identities of the bi-periodic k-Pell sequences like Catalan, Cassini and D'Ocagne's identities as well as some related summation formulas.

1 Introduction

For any natural number n and non-zero real numbers a and b, bi-periodic Fibonacci sequence was defined recursively by Edson and Yayenie [1] as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \quad \text{is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \quad \text{is odd,} \end{cases} \qquad n \geq 2$$

with initial condition $q_0 = 0, q_1 = 1$. Bi-periodic Lucas sequence was defined recursively by Bilgici [2] as

$$l_n = \begin{cases} al_{n-1} + l_{n-2}, & \text{if } n \quad \text{is even} \\ bl_{n-1} + l_{n-2}, & \text{if } n \quad \text{is odd,} \end{cases} \qquad n \geq 2$$

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with initial condition $l_0 = 2, l_1 = a$. In [3], Uygun and Owusu defined bi-periodic Jacobsthal sequence as

$$J_0 = 0, J_1 = 1, J_n = \begin{cases} aJ_{n-1} + 2J_{n-2}, & \text{if } n \quad \text{is even} \\ bJ_{n-1} + 2J_{n-2}, & \text{if } n \quad \text{is odd,} \end{cases} \quad n \geq 2.$$

Uygun and Karatas [4] defined bi-periodic Pell-Lucas sequence as

$$Q_0 = 2, Q_1 = 2a, Q_n = \begin{cases} 2aQ_{n-1} + Q_{n-2}, & \text{if } n \quad \text{is odd} \\ 2bQ_{n-1} + Q_{n-2}, & \text{if } n \quad \text{is even,} \end{cases} \quad n \geq 2.$$

In [1, 2, 3, 4], the Binet's formula, identities such as Catalan, Cassini and the D'Ocagne's as well as some related summation formulas were also given. In this paper, we define bi-periodic k-Pell sequence and some identities are also given.

2 Main results

Definition 2.1. *Let k be a positive real number. For any two non-zero real numbers a and b where ab does not contain any values belonging to the interval $[-k, 0]$, the bi-periodic k-Pell sequence, denoted by $\{P_{k,n}\}_{n=0}^{\infty}$, is defined recursively by*

$$P_{k,0} = 0, P_{k,1} = 1, P_{k,n} = \begin{cases} 2aP_{k,n-1} + kP_{k,n-2}, & \textit{if } n \textit{ is odd,} \\ 2bP_{k,n-1} + kP_{k,n-2}, & \textit{if } n \textit{ is even,} \end{cases} n \geq 2.$$

The first five elements of the bi-periodic k-Pell sequence are

$$P_{k,0} = 0, P_{k,1} = 1, P_{k,2} = 2b, P_{k,3} = 4ab + k, P_{k,4} = 4b(2ab + k).$$

When $a = b = k = 1$, we have a Pell sequence. If we set $a = b = 1$, for some positive real number k, we get a k-Pell sequence. If $k = 1$, for some non-zero real number a and b, we get a bi-periodic Pell sequence. From the above definition,we have the nonlinear quadratic equation for bi-periodic k-Pell sequence by

$$x^2 - 4abx - 4abk = 0 \tag{2.1}$$

with roots α and β defined by

$$\alpha = 2\left(ab + \sqrt{ab(ab+k)}\right), \beta = 2\left(ab - \sqrt{ab(ab+k)}\right).$$

Note that $\alpha\beta = -4abk, (\alpha+k)(\beta+k) = k^2, ab(\alpha+k) = \frac{\alpha^2}{4}, ab(\beta+k) = \frac{\beta^2}{4}, \alpha(\beta+k) = -k\beta$.

Lemma 2.2. *For any positive real number k, the bi-periodic k-Pell sequence satisfies the following properties*

(i) $P_{k,2n} = (4ab+2k)P_{k,2n-2} - k^2 P_{k,2n-4,}$

(ii) $P_{k,2n+1} = (4ab+2k)P_{k,2n-1} - k^2 P_{k,2n-3}$.

Proof. By using definition 2.1,
(i) $P_{k,2n} = 4abP_{k,2n-2} + 2bkP_{k,2n-3} + kP_{k,2n-2} = (4ab+2k)P_{k,2n-2} - k^2 P_{k,2n-4}$
(ii)$P_{k,2n+1} = 4abP_{k,2n-1} + 2akP_{k,2n-2} + kP_{k,2n-1} = (4ab+2k)\,P_{k,2n-1} - k^2 P_{k,2n-3}$. □

Theorem 2.3. *The generating function for the bi-periodic k-Pell sequence is given by*

$$P(x) = \frac{x(1+2bx-kx^2)}{1-(4ab+2k)x^2+k^2x^4}\ . \tag{2.2}$$

Proof. The generating function $P(x) = \sum_{m=0}^{\infty} P_{k,2m}x^{2m} + \sum_{m=0}^{\infty} P_{k,2m+1}x^{2m+1}$ is divided into two parts (even and odd). The even and odd parts of the series are denoted by $P_0(x)$ and $P_1(x)$, respectively. We have $P_0(x) = 2bx^2 + \sum_{m=2}^{\infty} P_{k,2m}x^{2m}$ and $P_1(x) = x + (4ab+k)x^3 + \sum_{m=2}^{\infty} P_{k,2m+1}x^{2m+1}$. Then $P_0(x) = \frac{2bx^2}{1-(4ab+2k)x^2+k^2x^4}$ and $P_1(x) = \frac{x-kx^3}{1-(4ab+2k)x^2+k^2x^4}$.
Thus $P(x) = \frac{x(1+2bx-kx^2)}{1-(4ab+2k)x^2+k^2x^4}$. □

Theorem 2.4. *(Binet's Formula) For any positive real number k and natural number n, the bi-periodic k-Pell sequence is given by the following*

$$P_{k,n} = \frac{(2b)^{1-\xi(n)}}{(4ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right), \tag{2.3}$$

where the parity function

$$\xi(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

and

$$\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. The generating function $P(x)$ can be expressed using partial fraction decomposition as:

$$P(x) = \frac{1}{k^2}\left[\frac{\frac{\alpha k x}{\beta-\alpha} - \frac{\alpha^2}{2a(\beta-\alpha)}}{x^2 - \frac{(\alpha+k)}{k^2}} + \frac{\frac{-\beta k x}{\beta-\alpha} + \frac{\beta^2}{2a(\beta-\alpha)}}{x^2 - \frac{(\beta+k)}{k^2}}\right].$$

The Maclaurin series expression of the function $\frac{Ax+B}{x^2-C}$ is expressed in the form

$$\frac{Ax+B}{x^2-C} = -\sum_{n=0}^{\infty} AC^{-n-1}x^{2n+1} - \sum_{n=0}^{\infty} BC^{-n-1}x^{2n},$$

the generating function $P(x)$ can be expanded as:

$$\begin{aligned} P(x) =& \frac{1}{k^2}\left[-\sum_{n=0}^{\infty}\left(\frac{\alpha k}{\beta-\alpha}\right)\left(\frac{\alpha+k}{k^2}\right)^{-n-1} x^{2n+1} - \sum_{n=0}^{\infty}\left(\frac{-\alpha^2}{2a(\beta-\alpha)}\right)\left(\frac{\alpha+k}{k^2}\right)^{-n-1} x^{2n}\right. \\ & \left. -\sum_{n=0}^{\infty}\left(-\frac{\beta k}{\beta-\alpha}\right)\left(\frac{\beta+k}{k^2}\right)^{-n-1} x^{2n+1} - \sum_{n=0}^{\infty}\left(\frac{\beta^2}{2a(\beta-\alpha)}\right)\left(\frac{\beta+k}{k^2}\right)^{-n-1} x^{2n}\right]. \end{aligned}$$

By $(\alpha+k)(\beta+k) = k^2, ab(\alpha+k) = \frac{\alpha^2}{4}, ab(\beta+k) = \frac{\beta^2}{4}, \alpha(\beta+k) = -k\beta$ and the definition of $\xi(n)$, we have

$$\begin{aligned} P(x) =& \frac{1}{k^2}\left[\frac{k^2}{\beta-\alpha}\sum_{n=0}^{\infty}\left(\frac{\beta^{2n+1}-\alpha^{2n+1}}{(4ab)^n}\right)x^{2n+1} + \frac{2bk^2}{\beta-\alpha}\sum_{n=0}^{\infty}\left(\frac{\beta^{2n}-\alpha^{2n}}{(4ab)^n}\right)x^{2n}\right] \\ =& \sum_{n=0}^{\infty}\frac{(2b)^{1-\xi(n)}}{(4ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)x^n. \end{aligned}$$

Since $P(x) = \sum_{n=0}^{\infty} P_{k,n}x^n = \sum_{n=0}^{\infty} \frac{(2b)^{1-\xi(n)}}{(4ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)x^n$, then

$$P_{k,n} = \frac{(2b)^{1-\xi(n)}}{(4ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right).$$

□

Theorem 2.5. *The limit of every two consecutive terms of bi-periodic k-Pell sequences are*

$$\lim_{n\to\infty}\frac{P_{k,2n+1}}{P_{k,2n}}=\frac{\alpha}{2b},\quad \textit{and}\quad \lim_{n\to\infty}\frac{P_{k,2n}}{P_{k,2n-1}}=\frac{\alpha}{2a}. \tag{2.4}$$

Proof. By Binet's formula and $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n=0$, we have

$$\lim_{n\to\infty}\frac{P_{k,2n+1}}{P_{k,2n}}=\lim_{n\to\infty}\frac{\frac{(2b)^{1-\xi(2n+1)}}{(4ab)^{\lfloor\frac{2n+1}{2}\rfloor}}\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}\right)}{\frac{(2b)^{1-\xi(2n)}}{(4ab)^{\lfloor\frac{2n}{2}\rfloor}}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)}=\frac{1}{2b}\lim_{n\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}}{\frac{1}{\alpha}-\frac{1}{\alpha}\left(\frac{\beta}{\alpha}\right)^{2n}}=\frac{\alpha}{2b},$$

$$\lim_{n\to\infty}\frac{P_{k,2n}}{P_{k,2n-1}}=\lim_{n\to\infty}\frac{\frac{(2b)^{1-\xi(2n)}}{(4ab)^{\lfloor\frac{2n}{2}\rfloor}}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)}{\frac{(2b)^{1-\xi(2n-1)}}{(4ab)^{\lfloor\frac{2n-1}{2}\rfloor}}\left(\frac{\alpha^{2n-1}-\beta^{2n-1}}{\alpha-\beta}\right)}=\frac{1}{2a}\lim_{n\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n}}{\frac{1}{\alpha}-\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^{2n}}=\frac{\alpha}{2a}.$$

□

Theorem 2.6. *For any positive real number k and positive integer n,*

$$P_{k,-n}=\frac{(-1)^{n-1}}{k^n}P_{k,n}. \tag{2.5}$$

Proof. According to Theorem 2.4 and $\alpha\beta=-4abk$, we have

$$\begin{aligned}P_{k,-n}&=\frac{(2b)^{1-\xi(-n)}}{(4ab)^{\lfloor\frac{-n}{2}\rfloor}}\left(\frac{\alpha^{-n}-\beta^{-n}}{\alpha-\beta}\right)\\&=\frac{(2b)^{1-\xi(n)}}{(4ab)^{\lfloor\frac{-n}{2}\rfloor}(\alpha-\beta)}\left(\frac{(-1)(\alpha^n-\beta^n)}{(-4abk)^n}\right)\\&=\frac{(-1)^{n-1}}{k^n}\left(\frac{(2b)^{1-\xi(n)}}{(4ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\right)\\&=\frac{(-1)^{n-1}}{k^n}P_{k,n}.\end{aligned}$$

□

Theorem 2.7. *(Catalan's Identity) For any integers n and r with $n \geq r$, we have*

$$\frac{(2b)^{2\xi(r+1)}}{(4ab)^{\xi(n-r)-\xi(r)}}P_{k,n-r}P_{k,n+r} - \frac{(2b)^{2(\xi(r+1)-(-1)^{\xi(n)}\xi(r))}}{(4ab)^{\xi(n)-\xi(r)}}P_{k,n}^2 = (-1)^{n-r+1}k^{n-r}(2b)^{2(\xi(n+1)-(-1)^{\xi(n)}\xi(r))}P_{k,r}^2.$$

Proof. Since

$$\frac{(2b)^{2\xi(r+1)}}{(4ab)^{\xi(n-r)-\xi(r)}}P_{k,n-r}P_{k,n+r} = \frac{(2b)^{2(\xi(n+1)-(-1)^{\xi(n)}\xi(r)+\xi(r+1))}}{(4ab)^{n-\xi(r)}(\alpha-\beta)^2}\left(\alpha^{2n} - \alpha^{n-r}\beta^{n+r} - \alpha^{n+r}\beta^{n-r} + \beta^{2n}\right)$$

and

$$\frac{(2b)^{2(\xi(r+1)-(-1)^{\xi(n)}\xi(r))}}{(4ab)^{\xi(n)-\xi(r)}}P_{k,n}^2 = \frac{(2b)^{2(\xi(n+1)+\xi(r+1)-(-1)^{\xi(n)}\xi(r))}}{(4ab)^{n-\xi(r)}(\alpha-\beta)^2}\left(\alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n}\right),$$

we have

$$\frac{(2b)^{2\xi(r+1)}}{(4ab)^{\xi(n-r)-\xi(r)}}P_{k,n-r}P_{k,n+r} - \frac{(2b)^{2(\xi(r+1)-(-1)^{\xi(n)}\xi(r))}}{(4ab)^{\xi(n)-\xi(r)}}P_{k,n}^2 = (-1)^{n-r+1}k^{n-r}(2b)^{2(\xi(n+1)-(-1)^{\xi(n)}\xi(r))}P_{k,r}^2.$$

□

Corollary 2.8. *(Cassini's Identity) For any positive real number k and positive integer n,*

$$\frac{1}{(4ab)^{\xi(n-1)-1}}P_{k,n-1}P_{k,n+1} - \frac{(2b)^{-2(-1)^{\xi(n)}}}{(4ab)^{\xi(n)-1}}P_{k,n}^2 = (-1)^n k^{n-1}(2b)^{2(\xi(n+1)-(-1)^{\xi(n)})}.$$

Proof. For $r = 1$ in Catalan's identity, we have Cassini's identity. □

Theorem 2.9. *(D'Ocagne's Identity) For any positive real number k and positive integers m, n with $m \geq n$,*

$$\frac{(2b)^{\xi(m+2)+\xi(n+1)+\xi(m-n+1)}}{(4ab)^{\frac{-n+\xi(m)-\xi(n)-\xi(m-n)}{2}}}P_{k,m}P_{k,n+1} - \frac{(2b)^{\xi(m+1)+\xi(n+2)+\xi(m-n+1)}}{(4ab)^{\frac{-n-\xi(m)+\xi(n)-\xi(m-n)}{2}}}P_{k,m+1}P_{k,n} = 4(-k)^n b^2(4ab)^{\frac{n}{2}}P_{k,m-n}.$$

Proof. By Binet's formula, $\left\lfloor\frac{n+1}{2}\right\rfloor = \dfrac{n+\xi(n)}{2}$ and $1-\xi(n) = \xi(n+1)$, and we have

$$\frac{(2b)^{\xi(m+2)+\xi(n+1)+\xi(m-n+1)}}{(4ab)^{\frac{-n+\xi(m)-\xi(n)-\xi(m-n)}{2}}}P_{k,m}P_{k,n+1} - \frac{(2b)^{\xi(m+1)+\xi(n+2)+\xi(m-n+1)}}{(4ab)^{\frac{-n-\xi(m)+\xi(n)-\xi(m-n)}{2}}}P_{k,m+1}P_{k,n}$$
$$= \frac{(2b)^{2+\xi(m-n+1)}}{(4ab)^{\frac{m-\xi(m-n)}{2}}(\alpha-\beta)^2}\left(\alpha^{m+n+1} - \alpha^m\beta^{n+1} - \alpha^{n+1}\beta^m + \beta^{m+n+1}\right)$$
$$- \frac{(2b)^{2+\xi(m-n+1)}}{(4ab)^{\frac{m-\xi(m-n)}{2}}(\alpha-\beta)^2}\left(\alpha^{m+n+1} - \alpha^{m+1}\beta^n - \alpha^n\beta^{m+1} + \beta^{m+n+1}\right)$$

$$
\begin{aligned}
&= \frac{(2b)^{2+\xi(m-n+1)}}{(4ab)^{\frac{m-\xi(m-n)}{2}}(\alpha-\beta)^2}\left(\alpha^{m+1}\beta^n+\alpha^n\beta^{m+1}-\alpha^m\beta^{n+1}-\alpha^{n+1}\beta^m\right)\\
&= (2b)^2(4ab)^{\frac{n}{2}}(-1)^n k^n \frac{(2b)^{1-\xi(m-n)}}{(4ab)^{\left\lfloor\frac{m-n}{2}\right\rfloor}}\left(\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}\right)\\
&= 4(-k)^n b^2(4ab)^{\frac{n}{2}}P_{k,m-n}.
\end{aligned}
$$

□

Theorem 2.10. *For any positive real number k and non-negative integer n, we have*

(i) $$\sum_{r=0}^{n}\binom{n}{r}(2b)^{\xi(r)}(4ab)^{\left\lfloor\frac{r}{2}\right\rfloor}k^{n-r}P_{k,r}=P_{k,2n},$$

(ii) $$\sum_{r=0}^{n}\binom{n}{r}(2b)^{\xi(r+1)}(4ab)^{\xi(r)+\left\lfloor\frac{r}{2}\right\rfloor}k^{n-r}P_{k,r+1}=2bP_{k,2n+1}.$$

Proof. To prove (i) and (ii), we use Theorem 2.4. □

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