International Journal of Mathematics and
Computer Science, **19**(2024), no. 1, 41–47

Comparison of the Mean Estimates of 2-Parameter Exponential Distribution by Multiple Criteria Decision Making Method

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(Received May 13, 2023, Accepted June 28, 2023,
Published August 31, 2023)

Abstract

In this paper we consider the estimators of the mean of 2-parameter exponential distribution. The estimators of mean are the class of invariant estimator and a class of shrinkage estimators. The performances of the estimators are compared based on the Multiple Criteria Decision Making (MCDM). Our purpose is to present the theorem of comparing these estimators. The result shows that shrinkage estimators $\hat{\mu}_{(1)}$ is the best estimators while $\hat{\mu}_{(2)}$ is the worst when sample size is greater than four.

1 Introduction

The random variable X has an exponential distribution and θ is a parameter, where $\theta > 0$ has a following probability density function as

$$f(x) = \frac{1}{\theta} exp(\frac{-x}{\theta}), x > 0, \theta > 0,$$

Key words and phrases: Multiple criteria decision making, mean estimates, 2-parameter exponential distribution.
AMS (MOS) Subject Classifications: 62C25, 62G05.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

where mean and variance are θ and θ^2, respectively. If we know that random variable X is greater than γ, where γ is any real number, then X has 2-parameter exponential distribution. The probability density function is formulated as

$$f(x) = \frac{1}{\theta}exp(\frac{-(x-\gamma)}{\theta}), x > \gamma, \theta > 0,$$

where θ is a scale parameter and γ is a location parameter. Mean and variance are $\mu = \gamma + \theta$ and θ^2, respectively.

Let $x_1, x_2, ..., x_n$ be a random sample of size n from an exponential population. In 1994, Kourouklis [6] proposed the estimator of μ in the class of invariant estimators is $\hat{\mu}_{MMSE} = x_{(1)} + \frac{n-1}{n}(\overline{x} - x_{(1)})$, $x_{(1)} = min\{x_i | i = 1, 2, ..., n\}$ and a class of shrinkage estimators for μ when given a prior estimate of the scale parameter θ_0 is $\hat{\mu}_{(p)} = x_{(1)} + \frac{n-1}{n}\hat{\theta}_{(p)}$, $\hat{\theta}_{(p)} = \theta_0 + \alpha(p) \cdot (\hat{\theta} - \theta_0)$ where $\alpha(p) = \frac{\Gamma(n-1-p)}{\Gamma(n-1-2p)(n-1)^p}, p \in (-\infty, \frac{1}{2}(n-1))$.

In this paper, we compare the estimators of mean in 2-parameter exponential distribution which are from the class of invariant estimator and a class of shrinkage estimators on the basis of mean squared errors by using Multiple Criteria Decision Making (MCDM) method.

2 Description of MCDM procedure

Multiple Criteria Decision making (MCDM) is a method to integrate the multiple risks $(x_{i1}, ..., x_{iN})$ for the i^{th} estimator into a single meaningful and overall risk factor (see [1,2,3,4,5]). The K estimators are then compared on the basis of these integrated risk factors. In the context of a discrete risk matrix $X = (x_{ij}) : K \times N$ where $x_{ij}\,'s$ represent risk of i^{th} estimator for j^{th} parameter point, and we need to compare K estimators simultaneously with respect to all the N parameter points. Integration of risks is done by defining an Ideal Row (IDR) with the smallest observed value for each column as

$$IDR = (\min_i x_{i1}, ..., \min_i x_{iN}) = (u_1, ..., u_N)$$

and a Negative-ideal Row *(NIDR)* with the largest observed value for each column as

$$NIDR = (\min_i x_{i1}, ..., \min_i x_{iN}) = (v_1, ..., v_N).$$

For any given row i, we now compute the distance of each row from Ideal row and from Negative Ideal row based on a suitably chosen norm. Under L_1-norm, we compute

$$\begin{aligned} L_1(i, IDR) &= \sum_{j=1}^{N} (x_{ij} - u_j)\ w_j \\ L_1(i, NIDR) &= \sum_{j=1}^{N} (v_j - x_{ij})\ w_j \end{aligned}$$

where $w_j\ 's$ are appropriate weights. The various rows are now compared based on an overall index computed as

$$L_1(Index_i) = \frac{L_1(i, IDR)}{L_1(i, IDR) + L_1(i, NIDR)}, \quad i = 1, ..., K.$$

A continuous version of this setup which is relevant for our problem would involve $x_{ij}\ 's$ representing risks or mean squared errors where the index j would vary continuously. In our problem of comparing $\hat{\mu}_{MMSE}$ and $\hat{\mu}_{(p)}$ when $p = -2, -1, 1, 2$ for estimation of μ. Therefore $x_i\ 's$ represent the mean squared errors of the five estimators which are functions of a real-valued r. In this case L_1-norm would be redefined as

$$\begin{aligned} L_1(i, IDR) &= \int_{\underline{r}}^{\overline{r}} (x_i(r) - u(r)) w(r) dr \\ L_1(i, NIDR) &= \int_{\underline{r}}^{\overline{r}} (v(r) - x_i(r)) w(r) dr \end{aligned}$$

where $u(r) = \min\limits_i \{x_i(r)\}$ and $v(r) = \max\limits_i \{x_i(r)\}, i = 1, 2, 3, 4, 5$ when $\underline{r} \leqslant r \leqslant \overline{r}$.

Furthermore of MCDM method, Lertprapai [7] proved a general result in the case of L_1-norm. Suppose $M_1, M_2, ..., M_K$ are some estimators to be compare with respect to their mean squared errors $MSE(M_i) = x_i, i = 1, 2, ..., K$, where $\underline{r} \leqslant r \leqslant \overline{r}$. Therefore, under L_1-norm, the i^{th} estimators is better than the j^{th} estimators if

$$L_1(Index_i) < L_1(Index_j) \tag{2.1}$$

that is

$$\int_{\underline{r}}^{\overline{r}} x_i(r) w(r) dr < \int_{\underline{r}}^{\overline{r}} x_j(r) w(r) dr, \tag{2.2}$$

where $w(r)$ is the weight function. Hence, we can use this inequality (2.2) to compare these estimators.

3 Mean squared errors

Refer to Kourouklis [6], the mean squared errors of $\hat{\mu}_{MMSE}$ and $\hat{\mu}_{(p)}$ when $p = -2, -1, 1, 2$ are given as the followings.

$$MSE(\hat{\mu}_{MMSE}) = \frac{\theta^2}{n^2} + \frac{(n-1)^2}{n^3}\theta^2 = \frac{\theta^2}{n^2}\left(\frac{n^2-n+1}{n}\right) \tag{3.1}$$

$$MSE(\hat{\mu}_{(p)}) = \frac{\theta^2}{n^2}\left[1 + (n-1)^2\left((1-\alpha(p))^2 r^2 + \frac{\alpha^2(p)}{(n-1)}\right)\right], \tag{3.2}$$

where $\alpha(p) = \dfrac{\Gamma(n-1-p)}{\Gamma(n-1-2p)(n-1)^p}, p \in (-\infty, \frac{1}{2}(n-1))$. Since (3.1) and (3.2) have common term $\dfrac{\theta^2}{n^2}$ which can be ignored so that mean squared errors are obtained in the function of r as

$$MSE(\hat{\mu}_{MMSE}) = \frac{n^2-n+1}{n} \tag{3.3}$$

and

$$MSE(\hat{\mu}_{(p)}) = 1 + (n-1)^2\left((1-\alpha(p))^2 r^2 + \frac{\alpha^2(p)}{(n-1)}\right) \tag{3.4}$$

where $\alpha(p) = \dfrac{\Gamma(n-1-p)}{\Gamma(n-1-2p)(n-1)^p}, r = \dfrac{\theta_0}{\theta} - 1$, θ_0 is prior estimate of θ. We consider when $p = -2, -1, 1, 2$.

4 Main result

Refer to Lertprapai [7], a general result in the case of L_1-norm to compare the estimators using inequality (2.2), we found that for $n \geqslant 4$ the ordering of μ is shown as the following theorem.

Theorem 4.1 If the estimators of μ in 2-parameter exponential distribution are the class of invariant estimators ($\hat{\mu}_{MMSE}$) and a class of shrinkage estimators ($\hat{\mu}_{(p)}, p = -2, -1, 1, 2$). Then based on mean squared errors, $\hat{\mu}_{(1)}$ is the best estimator while $\hat{\mu}_{(-1)}, \hat{\mu}_{MMSE}, \hat{\mu}_{(-2)}$ and $\hat{\mu}_{(2)}$ are lower in rank respectively for $n \geq 4$ under MCDM approach using L_1-norm and weight function $w(r) = 1$.

Proof. Since mean squared errors of $\hat{\mu}_{MMSE}$ and $\hat{\mu}_{(p)}, p = -2, -1, 1, 2$ are obtained from (3.3) and (3.4) respectively. Writing $x_1(r) = MSE(\hat{\mu}_{MMSE}), x_2(r) = MSE(\hat{\mu}_{(-2)}), x_3(r) = MSE(\hat{\mu}_{(-1)}),$

$x_4(r) = MSE(\hat{\mu}_{(1)}), x_5(r) = MSE(\hat{\mu}_{(2)})$.

From (2.2), the i^{th} estimator is better than the j^{th} estimator if

$$\int_{\underline{r}}^{\overline{r}} x_i(r)w(r)dr < \int_{\underline{r}}^{\overline{r}} x_j(r)w(r)dr.$$

Since $w(r) = 1$ and consider $-1 < r < 1$, we must then have

$$\int_{-1}^{1} x_i(r)dr < \int_{-1}^{1} x_j(r)dr.$$

For $n \geq 4$ we could show that

$$\int_{-1}^{1} x_4(r)dr < \int_{-1}^{1} x_2(r)dr < \int_{-1}^{1} x_3(r)dr < \int_{-1}^{1} x_1(r)dr < \int_{-1}^{1} x_5(r)dr.$$

Comparing each pair of inequality, thus we show in four cases as follow:

Case 1: $\int_{-1}^{1} x_4(r)dr < \int_{-1}^{1} x_2(r)dr$
We get

$$\int_{-1}^{1} \left[1 + (n-1)^2 \left((1-\alpha(1))^2 r^2 + \frac{\alpha^2(1)}{(n-1)}\right)\right] dr < \int_{-1}^{1} \left[1 + (n-1)^2 \left((1-\alpha(-2))^2 r^2 + \frac{\alpha^2(-2)}{(n-1)}\right)\right] dr$$

$$\frac{2(3n^2 - 11n + 20)}{(n-1)} < \frac{2(3n^4 + 7n^3 - 6n^2 + 27n + 5)}{(n+2)(n+1)^2},$$

which we may rewrite as

$$3n^4 - 4n^3 + 2n^2 - 100n - 45 > 0.$$

By mathematical induction, assume this inequality is ture for $n \geq 4$.

For $n = 4$,

$$3(4)^4 - 4(4)^3 + 2(4)^2 - 100(4) - 45 > 0.$$

This is ture. Assume the truth of the statement $3n^4 - 4n^3 + 2n^2 - 100n - 45 > 0$ for some n, now
$3(n+1)^4 - 4(n+1)^3 + 2(n+1)^2 - 100(n+1) - 45 = 3n^4 + 8n^3 + 8n^2 - 96n - 144$

$= (3n^4 - 4n^3 + 2n^2 - 100n - 45) + 12n^3 + 6n^2 + 4n - 99$ which is statement for $n+1$. Since $3n^4 - 4n^3 + 2n^2 - 100n - 45$ and $12n^3 + 6n^2 + 4n - 99$ are greater than zero for $n \geq 4$. So the statement is true for $n \geq 4$ and its truth for n implies its truth for $n+1$. Therefore it is true for all n.

Case 2: $\int\limits_{-1}^{1} x_2(r)dr < \int\limits_{-1}^{1} x_3(r)dr$

We obtain

$$\int\limits_{-1}^{1} \left[1 + (n-1)^2 \left((1-\alpha(-2))^2 r^2 + \frac{\alpha^2(-2)}{(n-1)}\right)\right] dr$$

$$< \int\limits_{-1}^{1} \left[1 + (n-1)^2 \left((1-\alpha(-1))^2 r^2 + \frac{\alpha^2(-1)}{(n-1)}\right)\right] dr$$

$$\frac{2(3n^4 + 7n^3 - 6n^2 + 27n + 5)}{(n+2)(n+1)^2} < \frac{2(3n^3 - 5n^2 + 7n - 2)}{3n^2},$$

which we may rewrite as

$2n^4 - 5n^3 + 3n^2 + n - 1 > 0,$

$n^3(2n-5) + 3n^2 + (n-1) > 0.$

Since $2n - 5 > 0$ and $n - 1 > 0$ for $n \geq 4$. Therefore, the inequality of this case is true.

Case 3: $\int\limits_{-1}^{1} x_3(r)dr < \int\limits_{-1}^{1} x_1(r)dr$

We get

$$\int\limits_{-1}^{1} \left[1 + (n-1)^2 \left((1-\alpha(-1))^2 r^2 + \frac{\alpha^2(-1)}{(n-1)}\right)\right] dr < \int\limits_{-1}^{1} \frac{n^2 - n + 1}{n} dr$$

$$\frac{2(3n^3 - 5n^2 + 7n - 2}{3n^2} < \frac{2(n^2 - n + 1)}{n},$$

which we may rewrite as

$n^2 - 2n + 1 > 0,$

$(n-1)^2 > 0$ which is true for all n.

Case 4: $\int\limits_{-1}^{1} x_1(r)dr < \int\limits_{-1}^{1} x_5(r)dr$

We get

$$\int_{-1}^{1} \frac{n^2-n+1}{n} dr < \int_{-1}^{1} \left[1+(n-1)^2\left((1-\alpha(2))^2 r^2 + \frac{\alpha^2(2)}{(n-1)}\right)\right] dr$$

$$\frac{2(n^2-n+1)}{n} < \frac{2(3n^4-2n^3+39n^2-444n+836)}{3(n-1)^3}$$

which we may rewrite as $10n^4+18n^3-423n^2+824n+3>0$. By mathematical induction, this inequality is true for $n \geq 4$. These complete the proof in four cases.

Acknowledgment. I would like to extend my appreciation to the Faculty of Science at Burapha University in Thailand for the support.

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